

Decoding of Threshold Functions Defined on the Integer Points of a Polytope¹

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Abstract—An algorithm of decoding a threshold function defined on the set of integer points of an arbitrary polytope is proposed. For any fixed dimension, the algorithm requires a polynomially bounded number of questions about the value of the function at a point. The set of all integer points of the polytope can be interpreted as the set of some objects divided into two categories. The algorithm can be useful in solving pattern recognition problems provided the dividing surface is a hyperplane and that the category can be defined from its arbitrary object.

1. INTRODUCTION

Let objects in a collection be characterized by a set of n numerical parameters (features) and divided into 2 categories (patterns). Then each object can be represented by a point and the whole collection by a domain in n -dimensional space. Each category is associated with its subdomain. Consider the case, where parameters of the objects are integers and the domain corresponding to the whole collection is bounded and represents the set of integer solutions to a given system of linear inequalities (the set of integer points of a polytope). Suppose the subdomains that represent different patterns can be divided by a hyperplane. As it was noted, for instance, in [1], this case is of "great practical importance." We assume that from an arbitrary object in the collection we can define which category it belongs to out of two categories and we also assume that *any* object in the whole collection is accessible for this definition at any time. These conditions set, we formulate the problem of defining the coefficients of the dividing hyperplane in the least possible number of examinations of the objects and during an acceptable time for intermediate calculations.

The threshold function of k -valued logic of n variables [2] is a mapping of a hypercube $\{0, 1, \dots, k-1\}^n$ into the set $\{0, 1\}$ such that there exists a hyperplane separating the set of points where the function is 0 from the set of points where the function is 1.

The problem of decoding threshold functions of k -valued logic is considered in [3]. By the decoding algorithm, we mean a procedure of finding the coefficients of the dividing hyperplane by means of questions about the value of the function at a point. It is shown in [3] that for any fixed n there exists a polynomial algo-

rithm of decoding the threshold function of k -valued logic, i.e., the number of questions about the value of the function and the number of necessary operations are limited in this algorithm to some polynomials of $\log k$. As it was shown in [4], the algorithm described in [3] requires

$$\kappa = O((\log k)^{n + \lfloor n/2 \rfloor (n-1)})$$

questions about the value of the function at the point ($\log \alpha$ is understood henceforth as $\log_2 \alpha$ and it is assumed in formulas using the upper-case O that the dimensionality of the space n is fixed).

A class of examples is investigated in [3] to demonstrate that an algorithm which would guarantee decoding any threshold function of k -valued logic of n variables after a number of questions bounded polynomially in n and $\log k$ does not exist.

We discuss below the following generalization of the concept of threshold function, which can expand considerably the application of the problem. Let the definition set of the function f be the set of integer points of a n -dimensional polytope P :

$$f: P \cap Z \rightarrow \{0, 1\}.$$

We again call f the threshold function if there exists a hyperplane in R^n , which separates the set of zeros of f from the set of units. Let P be defined by the system m of linear inequalities with integer-valued coefficients that do not exceed α by an absolute value. We propose here an algorithm A_0 that decodes for any fixed n an arbitrary threshold function defined on $P \cap Z$ during the time bounded polynomially in m and $\log \alpha$ using $O(m^{\lfloor n/2 \rfloor} \log^n \alpha)$ questions about the value of the function at a point. Note that for the case of threshold functions of k -valued logic the similar result ($O(\log^n k)$ questions are sufficient for decoding) was obtained in [5].

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2. DEFINITION AND NOTATION

Let the polytope P be given by the system of m linear inequalities with integer-valued coefficients that do not exceed α . by an absolute value. We denote the set of the integer points of the polytope P by $M(P)$. Suppose

$$f: M(P) \longrightarrow \{0, 1\}.$$

$M_0(f)$ and $M_1(f)$ are the sets of points x where f is zero or one, respectively, that is

$$M_v(f) = \{x \in M(P) | f(x) = v\} \quad (v = 0, 1);$$

N_v is the set of extreme points of a convex envelope of the set M_v . The function $f(x)$ is the *threshold* one if there are real numbers a_i ($i = 0, 1, \dots, n$) such that

$$M_0(f) = \{x = (x_1, \dots, x_n) \in M(P) | \sum_{i=1}^n a_i x_i \leq a_0\}. \quad (1)$$

The inequality

$$\sum_{i=1}^n a_i x_i \leq a_0 \quad (2)$$

is called the *threshold inequality* of f . The set of all threshold functions defined on $M(P)$ is denoted by $F(P)$. In what follows, we need a subset $F_+(P)$ of the set $F(P)$. $F_+(P)$ includes those and only those functions in $F(P)$ for which there exists the threshold inequality (2) with the coefficient $a_0 > 0$.

Let an *oracle* allowing to define $f(x)$ from an arbitrary point $x \in M(P)$ be associated with each function $f \in F(P)$. By *decoding* the function $f \in F(P)$ we mean the procedure of using the oracle to find numbers a_0, \dots, a_n such that the equality (1) is satisfied. Following [3], we call the threshold function decoding algorithm A the *quasipolynomial* one if for any $f \in F(P)$ with a fixed n the number $\kappa(A)$ of calls for the oracle and the number $\rho(A)$ of arithmetic operations required are bounded from above to polynomials in m and $\log \alpha$.

3. AUXILIARY RESULTS

Lemma 1 (see, for example, [6]). No component of any integer-valued vector in P exceeds $(n + 1)^{(n+3)/2} \alpha^{n+1}$ by an absolute value.

Each function $f \in F(P)$ in $(n + 2)$ -dimensional vector space is associated with the cone $K(f)$ of *dividing functionals* (a_0, \dots, a_n, τ) , which describes the follow-

ing system of linear inequalities (compare [2]):

$$\begin{cases} \sum_{i=1}^n a_i x_i \leq a_0 & \text{for all } x = (x_1, \dots, x_n) \in M_0(f) \\ \sum_{i=1}^n a_i x_i \geq a_0 + \tau & \text{for all } x = (x_1, \dots, x_n) \in M_1(f) \\ \tau \geq 0. \end{cases} \quad (3)$$

Any solution (a_0, \dots, a_n, τ) of this system defines the threshold inequality (2) for f with the threshold value $\tau > 0$. The opposite is also true: the coefficients (a_0, \dots, a_n) of any threshold inequality of the function $f \in F(P)$ satisfy the system (3) for a positive value of the threshold τ . Clearly, the system (3) is equivalent to the system

$$\begin{cases} \sum_{i=1}^n a_i x_i \leq a_0 & \text{for all } x = (x_1, \dots, x_n) \in N_0(f) \\ \sum_{i=1}^n a_i x_i \geq a_0 + \tau & \text{for all } x = (x_1, \dots, x_n) \in N_1(f) \\ \tau \geq 0. \end{cases} \quad (4)$$

Lemma 2 (see [3]). For any function $f \in F(P)$ there exists the threshold inequality (2) such that its coefficients (a_0, \dots, a_n) are integer-valued and

$$|a_i| \leq ((n + 1)^{(n+4)/2} \alpha^{n+1})^{n+1} \quad (i = 0, \dots, n). \quad (5)$$

Proof:

The theory of linear inequalities (see, for example, [6]) implies that there exists in $K(f)$ a system of vectors $b_1, \dots, b_s \in \mathbb{Z}^{n+2}$ such that any $b \in K(f)$ is their linear combination with nonnegative coefficients and for any $i = 1, 2, \dots, s$ there is such a subsystem that consists of $(n + 1)$ inequalities of (3) and becomes an equality on b_i . Thus, the vector b_i can be chosen so that its j -th coordinate b_{ij} coincides with an accuracy up to a sign with the determinant of the submatrix of the order $(n + 1)$ of a matrix composed of coefficients from (3). Hadamard inequality and the estimate from Lemma 1 yield:

$$\begin{aligned} |b_{ij}| &\leq (n + 1)^{(n+1)/2} ((n + 1)^{(n+3)/2} \alpha^{n+1})^{n+1} \\ &= (n + 1)^{(n+1)/2 + (n+3)(n+1)/2} \alpha^{(n+1)^2} \\ &= (n + 1)^{(n+1)(n+4)/2} \alpha^{(n+1)^2} \end{aligned}$$

To conclude the proof, we only note that if there exists a rational solution (a_0, \dots, a_n, τ) of (3) for $\tau > 0$, there also exists an integer-valued solution that meets this requirement. The lemma is proved.

Consider now the category $F_+(P)$. Clearly, for any $f \in F_+(P)$ there exists a threshold of the form

$$\sum_{i=1}^n a_i x_i \leq 1, \tag{6}$$

where, as follows from Lemma 2, the coefficients a_i can be made rational and not superior modulo values from the right-hand side of the inequality (5). Considering the above, we write (3) as

$$\begin{cases} \sum_{i=1}^n a_i x_i \leq & \text{for all } x = (x_1, \dots, x_n) \in M_0(f) \\ \sum_{i=1}^n a_i x_i > & \text{for all } x = (x_1, \dots, x_n) \in M_1(f). \end{cases} \tag{7}$$

In the space $R^n = \{(a_0, \dots, a_n)\}$ the closure of the set of the solutions to this system is a polyhedron $|W(f)|$ where any interior point gives coefficients of the threshold inequality (6) of the function f . We denote by $|W(f)|$ the volume of the polyhedron $W(f)$.

Lemma 3. For any function $f \in F_+(P)$

$$|W(f)| \geq (n+1)^{-n(n+3)(n+5)/2 - n} \alpha^{-n(n+1)(n+2)}. \tag{8}$$

Proof:

Let (a_0, \dots, a_n) be the coefficients whose existence was asserted by Lemma 2. Suppose now $w = (w_1, \dots, w_n)$, where $w_i = 2a_i / (2a_0 - 1)$ ($i = 1, \dots, n$). Let us show that not only $w \in W(f)$ but even the whole domain

$$W' \equiv \prod_{i=1}^n [w_i - 1/2(a_0 + 1)nk, w_i + 1/2(a_0 + 1)nk] \subseteq W(f),$$

where $k = (n+1)^{(n+3)/2} \alpha^{n+1}$, and $\prod_{i=1}^n \Gamma_i$ denotes the Cartesian product $\Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_n$. For this purpose we consider the point $u = (u_1, \dots, u_n) \in R^n$ such that $u_i = w_i + \varepsilon_i$ ($i = 1, \dots, n$) and $\varepsilon_i \in [-1/2(a_0 + 1)nk; 1/2(a_0 + 1)nk]$.

a) Let $x = (x_1, \dots, x_n) \in M_0(f)$, that is $\sum_{i=1}^n x_i a_i \leq a_0$. Then

$$\begin{aligned} \sum_{i=1}^n u_i x_i &= \sum_{i=1}^n \frac{2a_i}{2a_0 + 1} x_i + \sum_{i=1}^n \varepsilon_i x_i \\ &\leq \frac{2a_0}{2a_0 + 1} + \frac{kn}{2(a_0 + 1)kn} \\ &= 1 - \frac{1}{2a_0 + 1} + \frac{1}{2a_0 + 2} < \end{aligned}$$

b) Let $x = (x_1, \dots, x_n) \in M_1(f)$, that is $\sum_{i=1}^n x_i a_i \geq a_0 + 1$. Then

$$\begin{aligned} \sum_{i=1}^n u_i x_i &= \sum_{i=1}^n \frac{2a_i}{2a_0 + 1} x_i + \sum_{i=1}^n \varepsilon_i x_i \\ &\geq \frac{2}{2a_0 + 1} \sum_{i=1}^n x_i a_i + \sum_{i=1}^n \varepsilon_i x_i \\ &\geq \frac{2(a_0 + 1)}{2a_0 + 1} - \frac{kn}{2(a_0 + 1)kn} \\ &= 1 + \frac{1}{2a_0 + 1} - \frac{1}{2a_0 + 2} > 1 \end{aligned}$$

Thus,

$$\begin{aligned} |W(f)| &\geq \frac{1}{(a_0 + 1)^n n^n k^n} = (a_0 + 1)^{-n} n^{-n} \\ &\quad \times (n+1)^{-n(n+3)/2} \alpha^{-n(n+1)} \end{aligned}$$

and from Lemma 2:

$$|W(f)| \geq (n+1)^{-n - n(n+3)(n+5)/2} \alpha^{-n(n+1)(n+2)}.$$

The lemma is proved.

Remark. We denote by V the right-hand side of the inequality (5):

$$V = ((n+1)^{(n+4)/2} \alpha^{n+1})^{n+1}.$$

After simple calculations it is easy to see that

$$W' \subseteq W(f) \cap \{a = (a_1, \dots, a_n) \in R^n \mid |a_i| \leq 3V\}$$

and so the volume of the domain $W(f) \cap \{a = (a_1, \dots, a_n) \in R^n \mid |a_i| \leq 3V\}$ is not smaller than the value in the right-hand side of (8) either.

4. THE ALGORITHM

This section describes and substantiates the quasipolynomial algorithm A_0 of decoding the function $f \in F(P)$. We again impose an additional requirement: $f \in F_+(P)$. The algorithm A_0' described below decodes under this assumption which is easy to get rid of, as we will demonstrate later on.

To decode f , it is sufficient to find an arbitrary point $a = (a_1, \dots, a_n) \in W(f)$. The algorithm A_0' produces successively hypotheses $a^{(1)}, a^{(2)}, \dots, a^{(s)}, \dots$ about the vector a . Every time a hypothesis $a^{(s)}$ is verified by a series of calls for the oracle. If the hypothesis is correct (that is $a^{(s)} \in W(f)$), the algorithm A_0' finishes the work. Otherwise the results of the verification—values of the function at several new points in $M(P)$ —are used. These points and the values at them give the coefficients of the inequalities which any point in $W(f)$ should satisfy.

The algorithm starts searching the point $a \in W(f)$ from a suspicious domain

$$W_0 = \{w = (w_1, \dots, w_n) \mid |w_i| \leq 3V, (i = 1, \dots, n)\}$$

gradually reducing its volume and producing a sequence of embedded polytopes: $W_0 \subseteq W_1 \subseteq \dots \subseteq W_s \subseteq \dots$. The verification of the next hypothesis $a^{(s)}$ either finishes the work of the algorithm or adds new inequalities to the ones that describe W_s . The point $a^{(s)}$, however, does not satisfy these new inequalities. Hence, to guarantee fast decoding (or, to be exact, to reduce quickly the volumes of the polytopes W_s) it is necessary to take the points from the center of the polytope W_s (see [7]) as the hypotheses $a^{(1)}, a^{(2)}, \dots, a^{(s)}$. By giving different definitions of the center of the polytope we obtain different decoding algorithms. The center of the polytope W_s is taken in this paper to be its centroid, i.e., the point

$$a^{(s)} = \frac{\int_{W_s} \xi d\xi}{\int_{W_s} d\xi}$$

The polytope W_s has a nonzero volume as implied by lemma 3 and so $a^{(s)}$ is its interior point.

Let us proceed now to the step-by-step description of the algorithm A'_0 .

Set $\tilde{M}_0 := \emptyset$ and $\tilde{M}_1 := \emptyset$ at the preliminary step.

At the $(3s - 2)$ th step ($s = 1, 2, \dots$) find the point $a^{(s)} = (a_1^{(s)}, a_2^{(s)}, \dots, a_n^{(s)})$ — the centroid of W_s described by the system of linear inequalities

$$\begin{cases} \sum_{i=1}^n w_i x_i \leq 1 & \text{for all } x = (x_1, \dots, x_n) \in M_0(f) \\ \sum_{i=1}^n w_i x_i \geq 1 & \text{for all } x = (x_1, \dots, x_n) \in M_1(f) \\ |w_i| \leq 3V & (i = 1, 2, \dots, n). \end{cases} \quad (9)$$

Step 3s - 1. Define the set $N_0^{(s)}$ of extreme points of the convex envelope of the set $M(P) \cap \{x = (x_1, x_2, \dots, x_n) \mid \sum_{i=1}^n x_i a_i^{(s)} \leq 1\}$ and the set $N_1^{(s)}$ of extreme points of the convex envelope of the set $M(P) \cap \{x = (x_1, x_2, \dots, x_n) \mid \sum_{i=1}^n x_i a_i^{(s)} > 1\}$.

Step 3s. Use the oracle to find $f(x)$ for each point x in $N_0^{(s)}$ and $N_1^{(s)}$. If for some x $f(x) = 1 - \nu$ and $x \in N_\nu$ ($\nu = 0, 1$), include x in the set \tilde{M}_ν . If there are no such points, i.e., no points were added either in \tilde{M}_0 or in \tilde{M}_1 at this step, stop.

Suppose all the procedures used by the above algorithm (finding the centroid, finding the sets of extreme points, etc.) can be performed in a finite number of steps. The following lemma shows then that if the algo-

rithm A'_0 finished the work, the function $f \in F_+(P)$ is decoded, i.e., the next hypothesis $a^{(s)} \in W(f)$.

Lemma 4. Let $g, f \in F(P)$. If for any point $x \in N_0(g)$ $f(x) = 0$ and for any point $x \in N_1(g)$ $f(x) = 1$, then $g \equiv f$.

The proof clearly follows from the equivalence of the systems (3) and (4).

To estimate the number of calls for the oracle in A'_0 we use first the following geometric lemma ($|G|$ designates the volume of the domain G).

Lemma 5 (see, for example, [8]). Let G be a bounded closed convex body in R^n and G_+ and G_- be the parts into which G is divided by the hyperplane passing through the centroid of the body. Then

$$\max\{|G_+|, |G_-|\} \leq \left(1 - \left(\frac{n}{n+1}\right)^n\right) |G| \leq \frac{e-1}{e} |G|,$$

where e is the base of natural logarithms.

This lemma shows that the choice of the centroid of W_s as the next hypothesis $a^{(s)}$ really guarantees a significant reduction at least by a factor of $(e - 1)/e$ of the volume of the suspicious domain: the polytope W_{s+1} is certainly convex and $a^{(s)} \notin W_{s+1}$ or (in the last resort) $a^{(s)}$ is in the boundary of W_{s+1} and so W_{s+1} is contained entirely in one of the parts into which W_s is divided by the hyperplane passing through $a^{(s)}$. It follows from there and from the remark to Lemma 3 that any function $f \in F_+(P)$ will be decoded in no more than S_{\max} hypotheses of the form " $a^{(s)} \in W(f)$?" where

$$S_{\max} = \frac{\log(|W_0|/|W(f)|)}{\log e/(e-1)} + 1$$

We use the estimates (5) and (8) to find that the necessary number of hypotheses is bounded from above to a linear function of $\log \alpha$. And the number of questions to the oracle for testing one hypothesis, as the following lemma shows, is $O(m^{\lfloor n/2 \rfloor} \log^{n-1} \alpha)$.

Lemma 6 [9]. If the polyhedron $Q \subseteq R^n$ is set by the system l of linear inequalities with integer-valued coefficients that do not exceed β by an absolute value, the number of vertices in the convex envelope of the set $Q \cap Z^n$ is $O(l^{\lfloor n/2 \rfloor} \log^{n-1}(\beta + 1))$.

Lemmas 2 and 6 imply that the power $|N_\nu^{(s)}|$ of the set $N_\nu^{(s)}$ ($\nu = 0, 1$) constructed at step $(3s - 1)$ of the algorithm is $O(m^{\lfloor n/2 \rfloor} \log^{n-1} \alpha)$. Taking into account the estimate of the number of hypotheses " $a^{(s)} \in W(f)$?", we get

$$\kappa(A'_0) = O(m^{\lfloor n/2 \rfloor} \log^n \alpha)$$

To estimate $\rho(A'_0)$, we give some more definitions (see, for example, [6]). The length of the rational number

$\beta = p/q$, where p and q are coprimes, and the length of the rational vector $b = (\beta_1, \dots, \beta_n)$ are, respectively,

$$\begin{aligned} \text{size}(\beta) &= 1 + \log_2(|p| + 1) + \log_2(|q| + 1), \\ \text{size}(b) &= n + \text{size}(\beta_1) + \dots + \text{size}(\beta_n). \end{aligned}$$

Lemma 7. Let the polytope $Q \subset R^n$ be defined by the system l of linear inequalities with integer-valued coefficients that do not exceed β by an absolute value. Then the centroid α of the polytope Q is a rational point of a length not greater than a polynomial in l and $\log \beta$ and there exists an algorithm of its searching, which is polynomial in l and $\log \beta$.

Proof:

Let us construct this algorithm. First we find the partition of the polytope Q into the simplexes S_i , i.e., we obtain the representation

$$Q = \bigcup_{i=1}^{\sigma} S_i, \tag{10}$$

such that S_i ($i = 1, \dots, \sigma$) is a simplex and the affine dimensions of any two different simplexes in this representation are less than n . The algorithms that can build the list of vertices of the simplexes S_1, \dots, S_{σ} from the system of inequalities describing the polytope Q during the time which is polynomially bounded in l and $\log \beta$ for a partition (10) were described in [10]. The centroid $z = (z_1, \dots, z_n)$ and the volume $|S|$ of the simplex S are known to be given, respectively, by

$$z_i = \frac{v_j^{(0)} + v_j^{(1)} + \dots + v_j^{(n)}}{n + 1}$$

$$|S| = \frac{1}{n!} \begin{vmatrix} v_1^{(0)} & v_n^{(0)} \\ v_1^{(1)} & v_n^{(1)} \\ \dots & \dots \\ v_1^{(n)} & v_n^{(n)} \end{vmatrix}$$

where $v_{\mu} = (v_1^{(\mu)}, v_2^{(\mu)}, \dots, v_n^{(\mu)})$ ($\mu = 0, 1, \dots, n$) are the vertices of S , and to determine the volume we take the absolute value of the appropriate determinant. The coordinates of the centroid $a = (a_1, \dots, a_n)$ of the whole polytope Q are expressed then through the coordinates $z^{(i)} = (z_1^{(i)}, \dots, z_n^{(i)})$ of the centroids and volumes $|S_i|$ of the simplexes S_i as follows:

$$a_j = \frac{\sum_{i=1}^{\sigma} |S_i| z_j^{(i)}}{\sum_{i=1}^{\sigma} |S_i|}$$

The estimates of the lengths of the components of the centroids a and the computational cost of the algo-

rithm follow now from the above formulas. The lemma is proved.

The number of the inequalities in the system (9) that describes the polytope W_s can obviously exceed the total number of questions to the oracle only by $2n$. Absolute values of the coefficients of this system, as it follows from Lemma 2, are bounded from above by a polynomial in α with a fixed n . From Lemma 7, we have that the centroid $a^{(s)}$ of the polytope W_s at step $(3s - 2)$ of the algorithms A_0' can be found during the time that is bounded polynomially in m and $\log \alpha$. To find the sets $N_0^{(s)}$ and $N_1^{(s)}$ at step $(3s - 1)$ we use the algorithm of constructing extreme points of the convex envelope, polynomial for a fixed n , of the integer points of a polyhedron (see [11]). Due to the boundedness of the length of the vectors $a^{(s)}$ ($s = 1, 2, \dots$), polynomial in m and $\log \alpha$, the sets $N_0^{(s)}$ and $N_1^{(s)}$ will be found during the time that is bounded polynomially in m and $\log \alpha$ with a fixed n . The quasipolynomiality of the algorithm A_0' is proved.

What remains is to get rid of the assumption that $f \in F_+(P)$. Suppose now that $f \in F(P)$. We use the oracle to find the value of $f(x)$ in all vertices of the convex envelope of the set $M(P)$. If it turns out that for each such vertex $f(x) = v$, then $f(x) \equiv v$ ($v = 0, 1$). Otherwise, having selected any vertex v , where $f(v) = 0$ we transfer the origin of coordinates to the point v , i.e., we perform the transformation

$$x' = x - v.$$

Then P becomes a new polytope P' and the function f becomes f' :

$$f': P' \rightarrow \{0, 1\}.$$

Clearly, $f' \in F_+(P')$ and so we can apply the algorithm A_0' to decoding f' . Thus, we proved the following

Theorem 1. There exists the algorithm A_0 of decoding a threshold function in the category $F(P)$, for which the computational cost $\rho(A_0)$ is limited by a polynomial in m and $\log \alpha$, and the number $\kappa(A_0)$ of calls for the oracle is $O(m^{[n/2]} \log^n \alpha)$.

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