Integer program with bimodular matrix

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Abstract

Let A be an $m \times n$ integral matrix of rank n. We say that A is *bimodular* if the maximum of the absolute values of the $n \times n$ minors is at most 2. We give a polynomial time algorithm that finds an integer solution for system $Ax \leq b$. A polynomial time algorithm for integer program $max\{cx : \ Ax \leq b\}$ is constructed proceeding on some assumptions.

Key words: integer vertex; integer solution.

We use the following notation: if A is a matrix, A_i denotes its i - th row, $a_{i,j}$ is the entry at row i and column j, $\Delta_k(A)$ is the maximum of the absolute values of the $k \times k$ minors of A; if b is a vector then b_i denotes its i - th coordinate. Let $\lfloor \alpha \rfloor$ denote the largest integer less than or equal to α. The transpose of a matrix A is denoted by A^T. We say that $A \in \mathbb{Z}^{m \times n}$ is *bimodular* if $rank\ A = n$ and $\Delta_n(A) \leq 2$. By definition, put $S_Z = conv(S \cap \mathbb{Z}^n)$ for every $S \subseteq \mathbb{R}^n$. Let $M(A, b)$ be the set $\{x \in \mathbb{R}^n : \ Ax \leq b\}$.

Theorem 1 *If A is bimodular, $b \in \mathbb{Z}^n$, and $M(A, b)$ is full-dimensional, then $M_Z(A, b)$ is non-empty.*

Proof. We prove the statement by induction on n. If $n = 1$, then $M(A, b) \supseteq \{x \in \mathbb{R} : \ \beta - 1 \leq \alpha x \leq \beta\}$ for some $\beta \in \mathbb{Z}$ and $|\alpha| \in \{1, 2\}$. It is clear that either $\left\lfloor \frac{\beta}{\alpha} \right\rfloor$ or $\left\lfloor \frac{\beta-1}{\alpha} \right\rfloor$ belongs to $M_Z(A, b)$.

Now assume that $n > 1$. Since $rank\ A = n$ and $M(A, b) \neq \emptyset$, it follows that $M(A, b)$ has at least one vertex u (see e.g., [1], section 8.5.) If $u \in \mathbb{Z}^n$

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there is nothing to prove. Suppose $u \notin \mathbb{Z}^n$. Without loss of generality we can assume that u makes the first n inequalities into equalities that are $n-1$ dimensional faces of $M(A,b)$. Let d be the greatest common divisor(GCD) of $a_{1,1}, a_{1,2}, ..., a_{1,n}$. Denote by e_1 the first row of the identity matrix. We can find in polynomial time an unimodular matrix U such that $de_1 = A_1U$ (see e.g., [1], Corollary 5.3a.). If we replace x by Uy in $Ax \le b$, we get

$$\begin{pmatrix} d & 0 \\ h & \bar{A} \end{pmatrix} \begin{pmatrix} y_1 \\ \bar{y} \end{pmatrix} \le \begin{pmatrix} b_1 \\ \bar{b} \end{pmatrix}, \tag{1}$$

where h is a column vector, 0 is an all-zero row vector, $\bar{A}$ is a bimodular matrix with rank $n-1$, $\bar{b} = (b_2, .., b_m)^T$, $\bar{y} = (y_2, .., y_m)^T$.

Let T be the set of solutions of (1). We shall consider two cases.

First, let $b_1/d \in \mathbb{Z}$. The set $T \cap \{y : dy_1 = b_1\}$ has dimension $n-1$ and $\Delta(\bar{A}) \le 2/d$. By the induction hypothesis, there is an integral vector $\bar{y}^0 = (y_2^0, ..., y_n^0)$ such that $\bar{A}\bar{y}^0 \le \bar{b} - (b_1/d)h$. It follows that $U(b_1/d, y_2^0, ..., y_n^0)^T \in M_Z(A, b)$.

Suppose secondly that $d = 2$ and b_1 is odd. If $2y_1$ is unbounded from below on T, then $2w_1 \le b_1 - 1$ for some $w \in T$. Now assume that $2y_1$ is bounded from below on T. Since T is full-dimensional, we have $2v_1 = \min\{2y_1 : \ y \in T\} < b_1$, for some vertex v of T. Since $2v \in \mathbb{Z}^n$, it follows that $2v_1 \le b_1 - 1$.

So the set of solutions of the system

$$\bar{A}\bar{y} \le \bar{b} - \frac{b_1 - 1}{2} h \tag{2}$$

is non-empty. Since $\Delta(\bar{A}) = 1$, it follows that system (2) has an integer solution $(y_2^0, ..., y_n^0)$. Therefore, $U^{-1}((b_1 - 1)/2, y_2^0, ..., y_n^0)^T \in M_Z(A, b)$. $\square$

Note that this proof contains the effective algorithm that finds a point $x \in M_Z(A, b)$, where $M(A, b)$ is full dimensional(to find u we can use the algorithm from [2]). It is not difficult to see that x is the vertex of $M_Z(A, b)$.

If $M(A, b)$ is not full dimensional, then we can find $i_1, i_2, ..., i_s$ such that aff.hull $M(A, b) = \{x| \ A_{i_1}x = b_{i_1}, \ A_{i_2}x = b_{i_2}, \ ... \ A_{i_s}x = b_{i_s}\}$([1], Remark to Theorem 13.4). Next we can decide in polynomial time if $\{x \in \mathbb{Z}^n | \ A_{i_1}x = b_{i_1}, \ A_{i_2}x = b_{i_2}, \ ... \ A_{i_s}x = b_{i_s}\} = \emptyset$ or not([1], Corollary 5.3b). If not, then we can find in polynomial time the linearly independent integral vectors $h_0, h_1, ..., h_s$ such that $M_Z(A, b) \subseteq \{h_0 + y_1h_1 + y_2h_2 + ... + y_sh_s | y_1, y_2, ..., y_s \in$

$\mathbb{Z}\}$([1], Corollary 5.3c). It is not difficult to see, that the initial problem can be reduced to a like problem in s unknowns.

From now on we assume that A is bimodular, $b \in \mathbb{Z}^m$, and we simplify notation by writing $M \equiv M(A,b)$. Denote by $V(P)$ the vertex set of the polyhedra P. Let us now examine the set $V(M_Z)$.

Let u be a vertex of M, $I(u) = \{i : \sum_{j=1}^n a_{ij}u_j = b_i\}$, $N(u) = \{x : \sum_{j=1}^n a_{ij}x_j \leq b_i,\ i \in I(u)\}$.

Theorem 2 *Each vertex of $N_Z(u)$ lies on an edge of M.*

Proof. If $u \in \mathbb{Z}^n$ then u is the unique vertex of $N_Z(u)$ and the theorem holds.

Now assume that $u \notin \mathbb{Z}^n$. Let y be a vertex of $N_Z(u)$. Denote by A' the matrix obtained from A by omitting all rows $k \notin I(u)$. Let C be the cone $\{x :\ A'x \leq 0\}$. Since $A'y \leq A'u$, and since $A'y \neq A'u$ it follows that there exists an extremal ray r of C such that $\sum_{j=1}^n a_{kj}r_j = 0$ for all $k \in \{i \in I(u) : \sum_{j=1}^n a_{ij}y_j = b_i\}$. It is known that there exists an $(n-1)\times n$ submatrix H of A' such that $rank\ H = n-1$ and $Hr = 0$ [1]. Therefore, we can choose $r_j = \frac{1}{2}\sigma det\ H_j$ $(j = 1,2,...,n)$, where H_j is obtained from H by omitting the j-th column, $\sigma = \pm 1$. The matrix A is bimodular therefore

$$|\sum_{j=1}^n a_{ij}r_j| \leq 1 \ for\ all\ i \in \{1,2,...,m\}. \tag{3}$$

Let's assume that $r \in \mathbb{Z}^n$. Now we show that $y \pm r \in N_Z(u)$:

(a) for $i \in I(u)$ such that $\sum_{j=1}^n a_{ij}y_j = b_i$, we have $\sum_{j=1}^n a_{ij}r_j = 0$;

(b) for $i \in I(u)$ such that $\sum_{j=1}^n a_{ij}y_j < b_i$, from (3) we have $\sum_{j=1}^n a_{ij}(y_j \pm r_j) \leq b_i$.

Hence $y = \frac{1}{2}(y+r) + \frac{1}{2}(y-r)$, contradicting the fact that y is a vertex.

Thus we have $r \notin \mathbb{Z}^n$. Let B be an $n\times n$ submatrix of A' with nonzero determinant and let L be the lattice generated by the columns of the matrix B^{-1}. Note that $|detB| = 2$ (otherwise $u \in \mathbb{Z}^n$). Since $\det L = 1/2$, the index of sublattice $\mathbb{Z}^n$ in L is equal to 2. Therefore, L is divided into 2 classes: $\mathbb{Z}^n$ and $u+\mathbb{Z}^n$. Since $r \in u+\mathbb{Z}^n$, it follows that $u+r \in \mathbb{Z}^n$. Let us consider the vectors $p = u+r$ and $q = 2y-p$. From (3) it follows that $p \in M_Z$. For $i \in I(u)$ we have $\sum_{j=1}^n a_{ij}q_j = 2\sum_{j=1}^n a_{ij}y_j - \sum_{j=1}^n a_{ij}u_j - \sum_{j=1}^n a_{ij}r_j \leq \sum_{j=1}^n a_{ij}y_j - \sum_{j=1}^n a_{ij}r_j \leq b_i$ hence $q \in N_Z(u)$. Since y is a vertex and $y = \frac{1}{2}p + \frac{1}{2}q$, it follows that

$$y = p = u + r. \tag{4}$$

This implies that y belongs to an edge of M. $\square$

Corollary 3 $V(M_Z) = \bigcup\limits_{u\in V(M)} V(N_Z(u))$.

We now consider the problem of finding the maximum of the linear function $f = \sum\limits_j c_j x_j$ over M_Z.

Corollary 4 *If f achieves its maximum at vertex u of M, then $\max\limits_{x\in M_Z} f$ is achieved at some $y \in N_Z(u)$.*

Theorem 5 *If each $n \times n$ minor of A is not 0, then $\max\limits_{x\in M_Z} f$ can be found in polynomial time.*

Proof. With Khachiyan's method, we can find an optimum solution u for $\max\limits_{x\in M} f$ in polynomial time. Let $\max\limits_{x\in N_Z(u)} f$ be attained by vertex y in $N_Z(u)$. It follows from (4) that y belongs to one of the hyperplanes $\pi_1 = \{x : \ A_e x = b_e\}$ or $\pi_2 = \{x : \ A_e x = b_e - 1\}$, where $e \in I(u)$. We now prove that each vertex v of $N(u) \cap \pi_2$ remakes exactly n inequalities into equalities. Suppose to the contrary that there exists $k_1, k_2, ..., k_n \in I(u) \setminus \{e\}$ for which $A_{k_i} v = b_{k_i}$ $(i = 1, 2, ..., n)$. Since $\det(A^T_{k_1}\ A^T_{k_2}\ ...\ A^T_{k_n}) \neq 0$, it follows $v = u$. But this contradicts $A_e v = A_e u - 1$. So exactly n edges come out from v. Hence we can find in polynomial time a solution for $\max\limits_{x\in N_Z(u)\cap\pi_2} f$. In order to find $\max\limits_{x\in N_Z(u)\cap\pi_1} f$, we can solve the initial type problem with a lesser number of variables. $\square$

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References

[1] A. Schrijver, Theory of Linear and Integer Programming, WileyInterscience series in discrete mathematics. John Wiley & Sons, 1998.

[2] L.G.Khachiyan, Polynomial algorithms in the linear programming, U.S.S.R. Computational Mathematics and Mathematical Physics 20(1)(1980) 53-72.