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The Structure of Simple Sets in Z^3[1]

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Abstract—Polyhedrons in which every integral point belonging to them is a vertex are studied.

1. INTRODUCTION

A set M of points of an integral lattice is said to be simple if a polytope with vertices in M (every point of M must be vertex of this polytope) does not contain other points of the integral lattice. The power of a simple set in an n-dimensional integral lattice is not greater than 2^n(see [1]).

A simple set M is said to be maximal if it ceases to be simple on addition of a point of the integral lattice to it. Every simple set of power 2^n in an n-dimensional integral lattice is maximal. The converse may also hold, but its proof for the general case is not known to us. Its proof is known only for $n = 2$ and 3.

Definition 1. Parallel hyperplanes, each containing points of an integral lattice, are said to be adjacent if there are no points of the lattice between them.

Let M_1 and M_2 be simple sets lying in adjacent planes. Obviously, $M_1 \cup M_2$ is also a simple set. Below we shall show that every simple set in a three-dimensional space can be obtained in this way. A similar assertion does not hold for the four-dimensional space. The set of points $(0, 0, 0, 0)$, $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(-5, -12, -19, 29)$ cannot located on adjacent planes.

On the other hand, as is known [1], if a convex set M does not contain points of a lattice in its interior, then the maximal number of adjacent planes containing the intersection of M and an integral lattice is bounded above by a constant, depending on the dimension of the space.

Hence every simple set lies in not more than c_n parallel hyperplanes, where c_n is a constant dependent only on n. In other words, if a set of points $x_1, x_2, \dots, x_k$ is simple, then the system of inequalities $0 \le y(x_1 - x_i) \le c_n$, where $i = 2, \dots, k$, has a nonzero integral solution.

The parameter c_n is vital in application. In the sequel, we shall show that $c_3 = 2$.

2. THE STRUCTURE OF A SIMPLE SET OF A THREE-DIMENSIONAL LATTICE

First let us study the structure of the maximal simple set of a two-dimensional integral lattice.

Every simple set of power less than four in Z^2 is not maximal. To verify this assertion, let us consider a simple set consisting of three points. Clearly, these points do not lie on a straight line. Let us denote these points by x_1, x_2, and x_3. Consider the parallelogram with vertices at x_1, x_2, x_3, $x_3 + x_2 - x_1$. If a point $y \in Z^2$ is contained in this parallelogram, then it belongs to the triangle with vertices at x_2, x_3, $x_2 + x_3 - x_1$. But the point $x_2 + x_3 - y$ belongs to Z^2 and triangle with vertices x_1, x_2, x_3, contradicting the simplicity of the set $\{x_1, x_2, x_3\}$. Hence the simplicity of the set $\{x_1, x_2, x_3\}$ implies the simplicity of the three sets $\{x_1, x_2, x_3, x_2 + x_3 - x_1\}$, $\{x_1, x_2, x_3, x_1 + x_3 - x_2\}$, and $\{x_1, x_2, x_3, x_1 + x_2 - x_3\}$.

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Every simple set consisting of three points on a plane can be augmented to a maximal set only by three methods. Each of these methods yields a maximal set of four points, which are the vertices of some parallelogram.

We now study simple sets in Z^3.

Lemma 1. *If four points of a simple set M lie on a plane, then the set M belongs to adjacent planes.*

Lemma 2. *A tetrahedron with vertices in Z^3 and not containing other points of Z^3 lies between adjacent planes.*

Theorem. *A polyhedron M with vertices in Z^3 and not containing other points of Z^3 lies between adjacent planes.*

Corollary 1. *Every maximal simple set in Z^3 is the union of vertices of two parallelograms lying on adjacent planes.*

APPENDIX

Proof of Lemma 1. Let the vertices of a parallelogram be x_1, x_2, x_3, and x_4. Choosing a suitable coordinate system and changing the numeration of points, the coordinates of these point can be made such that $x_1 = (0, 0, 0)$, $x_2 = (1, 0, 0)$, $x_3 = (0, 1, 0)$, and $x_4 = (1, 1, 0)$.

In this coordinate system, the third component of all other points of M is ± 1. Indeed, assume the contrary, i.e., $x_5 = (a, b, c)$, $|c| > 1$. Let (d) be the fractional part of d and let $\{d\} = (1-(d))$. Take $\alpha = \min(\{a/|c|\}, \{b/|c|\})$, where $\beta = \{a/|c|\}-\alpha$, $\gamma = \{b/|c|\}-\alpha$, and $\delta = 1-\alpha-\beta-\gamma-1/|c|$. Here α, β, γ, and δ are nonnegative numbers and their sum is 1. The point $\delta x_1+\beta x_2+\gamma x_3+\alpha x_4+1/|c|x_5$ belongs to Z^3 and to the polyhedron with vertices in M, contradicting the simplicity of M.

If all points of M lie on one side of the plane $x_3 = 0$, then all points of M lie on two planes, either on $x_3 = 0$ and $x_3 = 1$, or on $x_3 = 0$ and $x_3 = -1$.

In M, let there exist points lying on different sides of the plane $x_3 = 0$. Two cases are possible. In case I, one point lies on one side of the plane $x_3 = 0$ and all other points (not lying on this plane) lie on the other side. In case II, every side of the plane contains two points each.

Let us examine case I in detail. The third base vector can be chosen such that $x_5 = (0, 0, 1)$ and all other points lie on the other side. Let us join x_5 with the points of M lying on the other side of the plane. The midpoints of the intervals thus constructed belong to the plane $x_3 = 0$ and can lie only on the lines $x_3 = 0$, $x_1 = 1/2$, and $x_3 = 0$, $x_2 = 1/2$. Indeed, the midpoints cannot belong to the set $K = \{(x, y)||x| \geq 1/2, |y| \geq 1/2\}$, because the point x_i is not a vertex of M in the contrary case. On the other hand, the midpoints of intervals joining the points in Z^3 belong to $(1/2)Z^3$. But the points of $(1/2)Z^3$ not belonging to K lie on the lines $x_3 = 0$, $x_1 = 1/2$ and $x_3 = 0$, $x_2 = 1/2$.

Let us assume that all midpoints of intervals lie on the line $x_3 = 0, x_1 = 1/2$. Then all points of M lies on two planes $x_1 = 1$ and $x_1 = 0$. Similarly, if the midpoints lie on the line $x_3 = 0, x_2 = 1/2$, then the points of M lie on the planes $x_2 = 1$ and $x_3 = 0$.

Now let us consider the case in which the midpoints of intervals lie on two lines. Without loss of generality, we can assume that the midpoints of intervals $[x_5, x_6]$ and $[x_5, x_7]$ lie on the lines $x_3 = 0, x_1 = 1/2$ and $x_3 = 0, x_2 = 1/2$, respectively. Then $x_6 = (1, a, -1)$ and $x_7 = (b, 1, -1)$. The inequalities $0 \leq a \leq 2$ and $0 \leq b \leq 2$ must both hold (otherwise, one of the points x_i, where $i = 1, 2, 3, 4$, is not a vertex of M). Let, for the sake of definiteness, $0 \leq a \leq 2$. If $a = 1$, then the midpoints of intervals lie on a line. Hence $a = 0$ or $a = 2$. Without loss of generality, we can take $a = 2$ (otherwise, we take a different coordinate system) and $x_6 = (1, 2, -1)$. The points $x_1, \ldots, x_7$ lie on the planes $x_2 + x_3 = 0$ and $x_2 + x_3 = 1$.

Let the point x_8 lie in the set M. The midpoint of the interval $[x_5, x_8]$ lies either on the line $x_3 = 0, x_2 = 1/2$, or is $(1/2, 0, 0)$. In the first case, M lies on two planes $x_2 + x_3 = 0$ and $x_2 + x_3 = 1$, whereas he second case is not possible, because the interval $[(1, 0, -1), (1, 2, -1)]$ contains a point with integral coordinates.

Let us consider the case in which both sides of the plane $x_3 = 0$ contain two points each. Let x_5 and x_6 lie on one side and x_7 and x_8 lie on the other side. Let us choose the third base vector such that $x_5 = (0, 0, 1)$. The midpoints of intervals (x_5, x_7), (x_5, x_8), (x_6, x_7), and (x_6, x_8) lie in the plane $x_3 = 0$, i.e., on the lines $x_3 = 0$, $x_1 = 1/2$ and $x_3 = 0$, $x_2 = 1/2$.

Let us assume that all midpoints of these intervals lie on the line $x_3 = 0$, $x_1 = 1/2$. Then all points of M lie on two planes $x_1 = 1$ and $x_1 = 0$. Similarly, if the midpoints lie on the line $x_3 = 0$, $x_2 = 1/2$, then the points of M lie on the planes $x_2 = 1$ and $x_3 = 0$.

Let us consider the case in which the midpoints of intervals lie on different lines. Note that the midpoints of three intervals cannot lie on the same line; otherwise all four points x_5, x_6, x_7, and x_8 lie on one plane and the midpoints of all four intervals lie on the same line. Consequently, the midpoints of two intervals lie on one line, whereas the midpoints of other two intervals lie on a different line. Since the points of the sets K_i cannot belong to $Conv(M)$, two midpoints are either $(1/2, 0, 0)$ and $(1/2, 1, 0)$, or $(0, 1/2, 0)$ and $(1, 1/2, 0)$. Changing the numeration of points such that the midpoints are $(1/2, 0, 0)$ and $(1/2, 1, 0)$, we find that the first of these points is the midpoint of the interval (x_5, x_7). Hence $x_7 = (1, 0, -1)$. The point $(1/2, 1, 0)$ cannot be the midpoint of the intervals (x_5, x_8) (then $x_8 = (1, 2, -1)$ and $(1/2)x_8 + (1/2)x_7 \in Z^3$) and (x_6, x_7) (then $x_6 = (0, 2, 1)$ and $(1/2)x_5 + (1/2)x_6 \in Z^3$). Consequently, this point is the midpoint of the interval (x_6, x_8). Let the midpoint of the interval (x_5, x_8) be $(a/2, 1/2, 0)$, where $a \in Z$. Then $x_8 = (a, 1, -1)$ and $x_6 = (1 - a, 1, 1)$. All points lie on two planes $x_2 = 0$ and $x_2 = 1$.

Proof of Lemma 2. We prove this lemma by *reductio ad absurdum*. Let T be the tetrahedron of least volume satisfying the conditions of the lemma for which the assertion of the lemma does not hold. Let the vertices of the tetrahedron T be x_1, x_2, x_3, and x_4. Without loss of generality, we can take $x_4 = 0$. Let $A = (x_1, x_2, x_3)$ and $\Delta = |\det\ A|$. If $\Delta = 1$, then the assertion of the lemma is obviously satisfied. Let $\Delta \geq 2$. Let S denote the set of all nonzero solutions of the system of equations $yA = 0(\Delta)$, whose components are nonnegative numbers, strictly less than Δ.

If S contains a y for which $y_1 + y_2 + y_3 \leq \Delta$, then the point $(y_1/\Delta)x_1 + (y_2/\Delta)x_2 + (y_3/\Delta)x_3 + (1 - (y_1 + y_2 + y_3)/\Delta)x_4$ belongs to $T \cap Z^3$. By the conditions of the lemma, the tetrahedron T does not contain any point of Z^3. Hence the inequality $y_1 + y_2 + y_3 > \Delta$ holds for any $y \in S$.

If S contains a y for which $y_1 + y_2 + y_3 \geq 2\Delta$, then S also contains a $z = -y(\Delta)$ for which $z_1 + z_2 + z_3 \leq \Delta$, which is impossible. Hence the inequality $y_1 + y_2 + y_3 < 2\Delta$ holds for every $y \in S$.

If S contains a y and a z such that $y_1 + y_2 + y_3 = z_1 + z_2 + z_3$, then S also contains a $v = y - z(\Delta)$ for which $v_1 + v_2 + v_3$ is either Δ or 2Δ, which is impossible by the above assumption.

The set S contains $\Delta - 1$ elements. Hence S contains a y for which $y_1 + y_2 + y_3 = 1 + \Delta$.

Let us take $x_5 = (y_1/\Delta)x_1 + (y_2/\Delta)x_2 + (y_3/\Delta)x_3 - (1/\Delta)x_4$. The determinant of the matrix $(x_1 - x_5, x_2 - x_5, x_3 - x_5)$ is 1 in absolute value. Introducing a coordinate system with center at the point x_5 and base $x_1 - x_5, x_2 - x_5, x_3 - x_5$, we find that $x_1 = (1, 0, 0)$, $x_2 = (0, 1, 0)$, $x_3 = (0, 0, 1)$, $x_4 = y$, and $x_5 = 0$. Let M be a polyhedron $Conv(x_1, \ldots, x_5)$. It can be represented by the union of tetrahedra T and $Conv(x_1, x_2, x_3, x_5)$. Since none of these tetrahedra contains points of Z^3, the polyhedron M also does not contain any point of Z^3, except its vertices. Express the polyhedron M as the union of tetrahedra $T_1 = Conv(x_2, x_3, x_4, x_5)$, $T_2 = Conv(x_1, x_3, x_4, x_5)$, and $T_3 = Conv(x_1, x_2, x_4, x_5)$. None of them contains points of Z^3 and each is less in volume than the tetrahedron T. Consequently, each of these tetrahedra lies between adjacent parallel planes. For the sake of definiteness, let $y_1 \geq y_2 \geq y_3$.

If y_3 is either 0 or 1, then T lies between adjacent parallel planes $x_3 = 0$ and $x_3 = 1$. Let $y_3 \geq 2$.

The condition that the tetrahedron T_1 lies between two adjacent parallel planes is equivalent to the existence of a nonzero solution $h \in Z^3$ for the system of inequalities $0 \leq h_2 \leq 1$, $0 \leq h_3 \leq 1$, and $0 \leq h_1y_1 + h_2y_2 + h_3y_3 \leq 1$. The solution of this system exists (since $y_1 \geq y_2 \geq y_3$) if $y_1 = y_2$ or $y_1 = y_2 + y_3$ or $y_1 + 1 = y_2 + y_3$.

If $y_1 = y_2$, then the point $(1/y_1)x_4 + (1 - y_3/y_4)x_3 + ((y_3 - 1)/y_1)x_5$ belongs to $M \cap Z^4$, which is impossible.

If $y_1 = y_2 + y_3$, then the condition that the tetrahedron T_2 lies between two adjacent parallel planes is equivalent (since $y_1 \geq y_2 \geq y_3$) to one of the equalities $y_1 - 2y_2 + y_3 = 0$ and $y_1 - 2y_2 + y_3 = 1$. In the first case, $y_1 = 3y_3$ and $y_2 = 2y_3$, and the point $(1/y_3)x_4 + (1 - 1/y_3)x_5$ belongs to $M \cap Z^3$. In the second case, $y_1 = 3y_3 - 1$ and $y_2 = 2y_3 - 1$. For $y_3 \geq 3$, we have $(1/y_3)x_1 + (1/y_3)x_2 + (1/y_3)x_4 + (1 - 3/y_3)x_5 \in M \cap Z^3$, and for $y_3 = 2$, we have $x_4 = (5, 3, 2)$ and $(1/3)x_1 + (1/3)x_3 + (1/3)x_4 \in M$. Hence this case is impossible.

If $y_1 + 1 = y_2 + y_3$, then the condition that the tetrahedron T_2 lies between two adjacent parallel planes is equivalent (since $y_1 \geq y_2 \geq y_3$) to one of the equalities: $y_3 = y_2$ and $y_1 - y_2 = 1$. In the first case, $y_1 = 2y_3 - 1$, $y_2 = y_3$, and $(1/y_3)x_1 + (1/y_3)x_4 + (1 - 2/y_3)x_5 \in M \cap Z^3$. In the second case, $y_1 = y_2 + 1$, $y_3 = 2$. Depending on the parity of y_2, the midpoint of one of the intervals, i.e., either $[x_1, x_4]$ or $[x_2, x_4]$, belongs to Z^3. Hence this case is also impossible.

Proof of Theorem. Let a tetrahedron T with vertices x_1, x_2, x_3, x_4 in M have the largest volume. Let us choose a coordinate system with center at x_1. By Lemma 2, this tetrahedron lies between two adjacent parallel planes, for example, $hx = 0$ and $hx = 1$.

Let x_5 be a vertex of M that is different from the vertices of the tetrahedron T. Let us express x_5 in terms of vectors x_2, x_3, and x_4 as $x_5 = \alpha_2x_2 + \alpha_3x_3 + \alpha_4x_4$. Since the volume of the tetrahedron T is maximal, the inequalities $|\alpha_2| \leq 1$, $|\alpha_3| \leq 1$, $|\alpha_4| \leq 1$, and $|\alpha_2 + \alpha_3 + \alpha_4 - 1| \leq 1$ hold.

If three vertices of the tetrahedron T lie on the plane $hx = 0$ or the plane $hx = 1$, then, without loss of generality, we can assume that the points x_1, x_2, and x_3 lie on the plane $hx = 0$. In this case, all other vertices of M can lie only on the planes $hx = 0, \pm 1$. Choosing the coordinate system (x_2, x_3, x_4), we find that $x_1 = (0, 0, 0)$, $x_2 = (1, 0, 0)$, $x_3 = (0, 1, 0)$, $x_4 = (0, 0, 1)$, and $h = (0, 0, 1)$. For other vertices of M, we have $(-1, 0, 1)$, $(-1, 1, 0)$, $(-1, 1, 1)$, $(0, -1, 1)$, $(0, 1, -1)$, $(0, 1, 1)$, $(1, -1, 0)$, $(1, -1, 1)$, $(1, 0, -1)$, $(1, 0, 1)$, $(1, 1, -1)$, and $(1, 1, 0)$. If at least one of the points $(-1, 1, 0)$, $(1, -1, 0)$, and $(1, 1, 0)$ belongs to M, then the assertion of the theorem follows from Lemma 1. If M does not contain the points $(0, 1, 1)$ and $(1, 0, 1)$, then the adjacent parallel planes on which M lies can be taken to be the planes $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 + x_3 = 1$.

If $(0, 1, 1) \in M$, then only the points $(-1, 0, 1)$, $(-1, 1, 1)$, $(1, 0, -1)$, and $(1, 0, 1)$ among the above points can be in M; otherwise, T will not be the maximal tetrahedron. Then the adjacent parallel planes on which M lies can be taken to be the planes $x_2 = 0$ and $x_2 = 1$.

If $(1, 0, 1) \in M$, then M may contain only the points $(0, -1, 1)$, $(0, 1, -1)$, $(0, 1, 1)$, and $(1, -1, 1)$. The planes $x_1 = 0$ and $x_1 = 1$ can be taken to be the planes between which M lies.

Let us assume that no three vertices of the tetrahedron T lie on the plane $hx = 0$ or the plane $hx = 1$. Let us also assume, for the sake of definiteness, that the vertices x_1 and x_2 lie on the plane $hx = 0$ and vertices x_3 and x_4 lie on the plane $hx = 1$. Note that $hx_5 = \alpha_3 + \alpha_4 \in Z$. If hx_5 is equal to 0 or 1 for all vertices of M, then the assertion of the theorem holds. The case $\alpha_3 + \alpha_4 = -2$ is impossible, because the inequalities $|\alpha_2| \leq 1$ and $|\alpha_2 - 3| \leq 1$ are incompatible. If $\alpha_3 + \alpha_4 = -1$, then the inequalities $|\alpha_2| \leq 1$ and $|\alpha_2 - 2| \leq 1$ have a unique solution $\alpha_2 = 1$, and $-\alpha_3x_3 - \alpha_4x_4 = x_2 - x - 5 \in Z^3 \cap M$. Therefore either $\alpha_3 = 0$ and $\alpha_4 = -1$, or $\alpha_3 = -1$ and $\alpha_4 = 0$. In the first case, the points x_5, x_1, x_2, and x_4 lie on one plane, whereas the points x_5, x_1,

x_2, and x_3 lie on one plane in the second case. In either case, the assertion of the theorem follows from Lemma 1.

Finally, let us consider the case in which $\alpha_3 + \alpha_4 = 2$. In this case, $\alpha_3 = \alpha_4 = 1$ and $-1 \leq \alpha_2 \leq 0$. Since $-\alpha_2 x_2 = (1 + \alpha_2)x_1 - \alpha_2 x_2 = x_5 - x_3 - x_4 \in M \cap Z^3$, we find that either $\alpha_2 = 0$ or $\alpha_2 = -1$. In the first case, the points x_1, x_3, x_4, and x_5 lie on one plane, whereas in the second case, the points x_2, x_3, x_4, and x_5 lie on one plane. In either case, the assertion of the theorem follows from Lemma 1.

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