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ARTICLE

Muonic–electronic negative hydrogen ion: circular states

N. Kryukov and E. Oks

Abstract: We studied a system consisting of a proton, a muon, and an electron (a μpe system), the muon and the electron being in circular states. We demonstrated that in this case, the muonic motion can represent a rapid subsystem while the electronic motion can represent a slow subsystem – the result that might seem counterintuitive. We used a classical analytical description to find the energy terms for the quasi molecule where the muon rotates around the axis connecting the immobile proton and the immobile electron (i.e., dependence of the energy of the muon on the distance between the proton and electron). We found that there is a double-degenerate energy term. We demonstrated that it corresponds to stable motion. We also conducted an analytical relativistic treatment of the muonic motion and found that the relativistic corrections are relatively small. Then we unfroze the slow subsystem and analysed a slow revolution of the axis connecting the proton and electron. We derived the condition required for the validity of the separation into rapid and slow subsystems. Finally, we showed that the spectral lines, emitted by the muon in the quasi molecule, μpe, experience a red shift compared to the corresponding spectral lines that would have been emitted by the muon in a muonic hydrogen atom (in the μp-subsystem). The relative values of this red shift, which is a "molecular" effect, are significantly greater than the resolution of available spectrometers and thus can be observed. Observing this red shift should be one of the ways to detect the formation of such muonic–electronic negative hydrogen ions.

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Résumé : Nous étudions un système constitué d'un proton, d'un muon et d'un électron (un système μpe), le muon et l'électron étant dans des états circulaires. Nous montrons que dans ce cas, le muon peut présenter un sous-système rapide et l'électron un sous-système lent — un résultat qui peut sembler contre-intuitif. Nous utilisons une description analytique classique pour trouver les termes d'énergie du muon en rotation autour d'un axe entre un proton et un électron immobiles (i.e. la dépendance de l'énergie du muon sur la distance entre le proton et l'électron). Nous trouvons un terme à double dégénérescence et démontrons qu'il correspond à un mouvement stable. Nous évaluons les corrections relativistes dans le mouvement du muon et les trouvons petites. Nous libérons alors le sous-système lent et analysons la *lente* révolution de l'axe pe. Nous obtenons les conditions de validité de la séparation du mouvement en sous-systèmes lent et rapide. Finalement, nous montrons que les lignes spectrales émises par le muon dans la quasi-molécule μpe montrent un déplacement vers le rouge par rapport aux lignes spectrales de l'atome μp (hydrogène muonique). Ceci est un effet *moléculaire* et la valeur du déplacement est significativement plus grande que la résolution des spectromètres existants et peut donc être observé. L'observation de ce déplacement vers le rouge serait une façon de détecter la formation de tels ions hydrogène négatifs muon–électron. [Traduit par la Rédaction]

1. Introduction

Studies of muonic atoms and molecules, where one of the electrons is substituted by the heavier lepton μ^-, have several applications. The first one is muon-catalyzed fusion (see, e.g., refs. 1–3 and references therein). When a muon replaces the electron either in the dde-molecule (D_2^+), which becomes the ddμ-molecule, or in the dte-molecule, which becomes the dtμ-molecule, the equilibrium internuclear distance becomes about 200 times smaller. At such small internuclear distances, fusion occurs with significant probability, which has been observed in ddμ or with an even higher rate in dtμ [1–3]. The second application is laser-control of nuclear processes. This has been discussed in the context of the interaction of muonic molecules with superintense laser fields [4]. Another application is a search for strongly interacting massive particles (SIMPs) proposed as dark matter candidates and as candidates for the lightest supersymmetric particle (see, e.g., [5] and references therein). SIMPs could bind to the nuclei of atoms, and would manifest as anomalously heavy isotopes of known elements. By greatly increasing the nuclear mass, the presence of a SIMP in the nucleus effectively eliminates the well-known reduced mass correction in a hydrogenic atom. Muonic atoms are better candidates (than electronic atoms) for observing this effect because the muon's much larger mass (compared to the electron) amplifies the reduced mass correction [5]. This may be detectable in astrophysical objects [5].

Another line of research is studies of the negative ion of hydrogen H^- (can also be denoted an epe-system (electron–proton–electron)), which constitute an important line of research in atomic physics and astrophysics. The epe-system has only one bound state — the ground state having a relatively small bound energy of approximately 0.75 eV. This epe-system exhibits rich physics. Correlations between the two electrons are strong already in the ground state. With long-range Coulomb interactions between all three pairs of particles, the dynamics is particularly subtle in a range of energies 2–3 eV on either side of the threshold for break-up into proton + electron + electron at infinity [6]. There are strong correlations in energy, angle, and spin degrees of freedom, so that perturbation theory and other similar methods fail [6]. Experimental studies of H^- provided a testing ground for the theory of correlated multielectron systems. Compared to the helium atom, the structure of H^- is even more strongly influenced by interelectron repulsion because the nuclear attraction is smaller for this system [7]. In addition to the preceding fundamental importance, the rich physics of H^- is also important in studies of the ionosphere's

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N. Kryukov and E. Oks. Physics Department, 206 Allison Lab., Auburn University, Auburn, AL 36849, USA.
Corresponding author: E. Oks (e-mail: goks@physics.auburn.edu).

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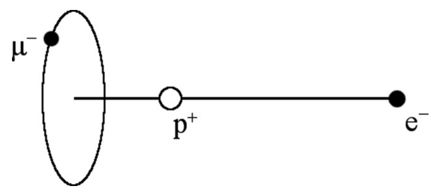


Fig. 1. A muon rotating in a circle perpendicular to and centered at the axis connecting the proton and the electron. The figure is not to scale; the actual muon–proton separation is much smaller than the electron–proton separation.

D-layer of Earth's atmosphere, the atmosphere of the Sun and other stars, and in development of particle accelerators [6].

In the present paper we combine the above two lines of research: studies of muonic atoms and molecules and studies of the negative hydrogen ion. Namely, we consider a muonic–electronic negative hydrogen ion (i.e., μpe-system). Specifically, we study a possibility of circular states in such system. We show that the muonic motion can represent a rapid subsystem, while the electronic motion can represent a slow subsystem — a result that might seem counterintuitive.

First, we find analytically classical energy terms for the rapid subsystem at the frozen slow subsystem (i.e., for the quasi molecule where the muon rotates around the axis connecting the immobile proton and the immobile electron). The meaning of classical energy terms is explained later. We demonstrate that the muonic motion is stable. We also conduct the analytical relativistic treatment of the muonic motion.

Then we unfreeze the slow subsystem and analyse a slow revolution of the axis connecting the proton and electron. We derive the condition required for the validity of the separation into rapid and slow subsystems.

Finally we show that the spectral lines, emitted by the muon in the quasi molecule μpe, experience a red shift compared to the corresponding spectral lines that would have been emitted by the muon in a muonic hydrogen atom (in the μp-subsystem). Observing this red shift should be one of the ways to detect the formation of such muonic–electronic negative hydrogen ions.

As for physical processes leading to the formation of muonic–electronic negative hydrogen ions, one of the processes could be the following:

$$e + \mu p \rightarrow \mu pe$$

(which sometimes might be followed by the decay $\mu pe \rightarrow \mu + pe$). Such formation of the μpe-systems was discussed [8], wherein these systems were called resonances. The theoretical approach based on the separation of rapid and slow subsystems requires in this case the muon to be in a state of high angular momentum. Luckily, the experimental methods to create muonic hydrogen atoms μp (necessary for the preceding reaction) lead to the muon being in a highly-excited state [9, 10]. It has been shown, in particular, that the distribution of the muon principal quantum number in muonic hydrogen atoms peaks at larger and larger values with the increase of the energy of the muon incident on electronic hydrogen atoms [11].

2. Analytical solution for classical energy terms of the rapid subsystem

We consider a quasi molecule where a muon rotates in a circle perpendicular to and centered at the axis connecting a proton and an electron — see Fig. 1. As we show later, in this configuration the muon may be considered the rapid subsystem while the proton and electron will be the slow subsystem, which essentially reduces the problem to the two stationary Coulomb center problem, where the effective stationary "nuclei" will be the proton and electron. The straight line connecting the proton and electron will be called the "internuclear" axis hereinafter. We use atomic units in this study.

Because of the difference of muon and electron masses, the muon–proton separation is much smaller than the electron–proton separation. Therefore, it should be expected that the spectral lines, emitted by this system, would be relatively close to the spectral lines emitted by muonic hydrogen atoms. In other words, the presence of the electron should result in a relatively small shift of the spectral lines (compared to muonic hydrogen atoms); however, this shift would be an important manifestation of the formation of the quasi molecule μpe.

A detailed classical analytical solution of the two stationary Coulomb center problem, where an electron revolves around nuclei of charges Z and Z', has been presented in refs. 12 and 13. We base our results in part on the results obtained therein.

The Hamiltonian of the rotating muon is

$$H = \frac{p_Z^2 + p_\rho^2 + \left(p_\varphi^2/\rho^2\right)}{2m} - Z(z^2 + \rho^2)^{-1/2} - Z'\left[(R - z)^2 + \rho^2\right]^{-1/2} \quad (1)$$

where m is the mass of the muon (in atomic units m = 206.768 274 6); Z and Z' are the charges of the effective nuclei (in our case, $Z = 1$ and $Z' = -1$); R is the distance between the effective nuclei, (ρ, φ, z) are the cylindrical coordinates, in which Z is at the origin and Z' is at $z = R$; and (p_ρ, p_φ, p_z) are the corresponding momenta of the muon.

Because φ is a cyclic coordinate, the corresponding momentum is conserved

$$|p_\varphi| = \text{const.} = L \quad (2)$$

With this substituted into (1), we obtain the Hamiltonian for the z- and ρ-motions

$$H_{z\rho} = \frac{p_z^2 + p_\rho^2}{2} + U_{\text{eff}}(z, \rho) \quad (3)$$

where an effective potential energy is

$$U_{\text{eff}}(z, \rho) = \frac{L^2}{2m\rho^2} - \frac{Z}{(z^2 + \rho^2)^{1/2}} - Z'\left[(R - z)^2 + \rho^2\right]^{-1/2} \quad (4)$$

Because in a circular state $p_z = p_\rho = 0$, the total energy $E(z, \rho) = U_{\text{eff}}(z, \rho)$.

With $Z = 1$, $Z' = -1$, and the scaled quantities

$$w = \frac{z}{R} \qquad v = \frac{\rho}{R} \qquad \varepsilon = -ER \qquad \ell = \frac{L}{(mR)^{1/2}} \qquad r = \frac{mR}{L^2} \quad (5)$$

we obtain the scaled energy, ε, of the muon

$$\varepsilon = (w^2 + v^2)^{-1/2} - \left[(1 - w)^2 + v^2\right]^{-1/2} - \frac{\ell^2}{2v^2} \quad (6)$$

The equilibrium condition with respect to the scaled coordinate w is $\partial\varepsilon/\partial w = 0$; the result can be brought to the form

$$\left[(1 - w)^2 + v^2\right]^{3/2}(w^2 + v^2)^{-3/2} = \frac{w - 1}{w} \quad (7)$$

Because the left-hand side of (7) is positive, the right-hand side must also be positive: $(w-1)/w > 0$. Consequently, the allowed ranges of w here are $-\infty < w < 0$ and $1 < w < +\infty$. This means that equilibrium positions of the center of the muon orbit could exist (judging only by the equilibrium with respect to w) either beyond the proton or beyond the electron, but there are no equilibrium positions between the proton and electron.

Solving (7) for v^2 and denoting $v^2 = p$, we obtain

$$p(w) = w^{2/3}(w-1)^{2/3}[w^{2/3} + (w-1)^{2/3}] \tag{8}$$

The equilibrium condition with respect to the scaled coordinate v is $\partial\varepsilon/\partial v = 0$, which yields

$$\ell^2 = p^2\{(w^2+p)^{-3/2} - [(1-w)^2+p]^{-3/2}\} \tag{9}$$

Because the left-hand side of (9) is positive, the right-hand side must be also positive. This entails the relation $w^2 + p < (1-w)^2 + p$, which simplifies to $2w - 1 < 0$, which requires $w < 1/2$.

Thus, the equilibrium with respect to both w and v is possible only in the range $-\infty < w < 0$, while in the second range, $1 < w < +\infty$ (derived from the equilibrium with respect to w only), there is no equilibrium with respect to v.

From the last two relations in (5), we find $r = 1/\ell^2$; thus

$$r = p^{-2}\{(w^2+p)^{-3/2} - [(1-w)^2+p]^{-3/2}\}^{-1} \tag{10}$$

where p is given by (8). Therefore, the quantity r in (10) is the scaled "internuclear" distance dependent on the scaled internuclear coordinate w.

Now we substitute the value of ℓ from (9), as well as the value of p from (8) into (6), obtaining $\varepsilon(w)$ — the scaled energy of the muon dependent on the scaled internuclear coordinate, w. Because $E = -\varepsilon/R$ and $R = rL^2/m$, then $E = -(m/L^2)\varepsilon_1$ where $\varepsilon_1 = \varepsilon/r$. The parametric dependence $\varepsilon_1(r)$ will yield the energy terms.

The form of the parametric dependence, $\varepsilon_1(r)$, can be significantly simplified by introducing a new parameter $\gamma = (1 - 1/w)^{1/3}$. The region $-\infty < w < 0$ corresponds to $1 < \gamma < \infty$. The parametric dependence will then have the following form:

$$\varepsilon_1(\gamma) = (1-\gamma)^4(1+\gamma^2)^2[2(1-\gamma+\gamma^2)^2(1+\gamma^2+\gamma^4)]^{-1} \tag{11}$$

$$r(\gamma) = (1+\gamma^2+\gamma^4)^{3/2}[\gamma(1+\gamma^2)^2]^{-1} \tag{12}$$

Classical energy terms given by the parametric dependence of the scaled energy $-\varepsilon_1 = (L^2/m)E$ on the scaled internuclear distance $r = (m/L^2)R$ are presented in Fig. 2.

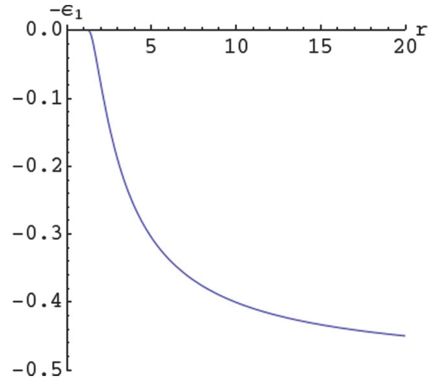
Figure 2 actually contains two coinciding energy terms: there is a double degeneracy with respect to the sign of the projection of the muon angular momentum on the internuclear axis. We remind the readers that L is the absolute value of this projection — in accordance with its definition in (2).

The minimum value of R, corresponding to the point where the term starts, can be found from (12). The term starts at $w = -\infty$, which corresponds to $\gamma = 1$; taking the value of (12) at this point, we find

$$R_{\min} = \frac{3^{3/2}}{4}\frac{L^2}{m} \tag{13}$$

With the value of $m = 206.768\,274\,6$, (13) yields $R = 0.006\,282\,58L^2$.

Fig. 2. Classical energy terms: the scaled energy, $-\varepsilon_1 = (L^2/m)E$, versus the scaled internuclear distance, $r = (m/L^2)R$.



The following note might be useful. The plot in Fig. 2 represents two degenerate classical energy terms of "the same symmetry". (In physics of diatomic molecules, the terminology "energy terms of the same symmetry" means the energy terms of the same projection of the angular momentum on the internuclear axis.) For a given R and L, the classical energy, E, takes only one *discrete* value. However, as L varies over a *continuous* set of values, so does the classical energy E (as it should be in classical physics).

The revolution frequency of the muon, Ω, is

$$\Omega = \frac{L}{m\rho^2} = \frac{L}{mR^2v^2} = \frac{L}{mR^2p} \tag{14}$$

in accordance with the previously introduced notation $p = v^2 = (\rho/R)^2$. Because $R = L^2r/m$ (see (5)), then (14) becomes $\Omega = (m/L^3)f$, where $f = 1/(pr^2)$. Using (12) for $r(\gamma)$ and (8) for $p(w)$ with the substitution $w = 1/(1-\gamma^3)$, where $\gamma > 1$, we finally obtain

$$\Omega = \frac{m}{L^3}f(\gamma) \qquad f(\gamma) = (1+\gamma^2)^3(1-\gamma^3)^2(1+\gamma^2+\gamma^4)^{-3} \tag{15}$$

where $f(\gamma)$ is the scaled muon revolution frequency. Figure 3 shows the scaled muon revolution frequency $f = (L^3/m)\Omega$ versus the scaled internuclear distance $r = (m/L^2)R$.

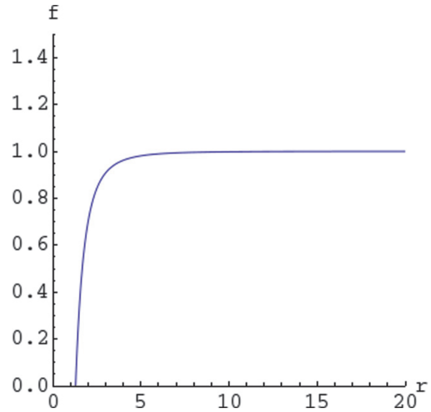
It is seen that for almost all values of the scaled internuclear distance $r = (m/L^2)R$, the scaled muon revolution frequency $f = (L^3/m)\Omega$ is very close to its maximum value, $f_{\max} = 1$, corresponding to large values of R. (The quantity $f_{\max}$ can be easily found from (15) given that large values of R correspond to $\gamma \gg 1$ and that this limit yields $f_{\max} = 1$.) In other words, for almost all values of R, the muon revolution frequency Ω is very close to its maximum value

$$\Omega_{\max} = \frac{m}{L^3} \tag{16}$$

In Sect. 3, we will compare the muon revolution frequency with the corresponding frequency of the electronic motion and derive the condition of validity of the separation into rapid and slow subsystems.

To analyse the stability of the muon motion, corresponding to the degenerate classical energy terms, while considering a classical circular motion of a charged particle (which was the electron in ref. 13) in the field of two stationary Coulomb centers, using the same notation as in the present paper, it was shown [13] that the frequencies of small oscillations of the scaled coordinates w and v of the circular orbit around its equilibrium position are given by

Fig. 3. The scaled muon revolution frequency, $f = (L^3/m)\Omega$, versus the scaled internuclear distance, $r = (m/L^2)R$.



$$\omega_{\pm} = \left[\frac{1}{1-w} \pm \frac{3w}{Q}\right]^{1/2}(w^2 + p)^{-3/4} \quad (17)$$

where

$$Q = (w^2 + p)^{1/2}[(1-w)^2 + p]^{1/2} \quad (18)$$

These oscillations are in the directions (w', v') obtained by rotating the (w, v) coordinates by the angle α

$$\delta w' = \delta w \cos\alpha + \delta v \sin\alpha \qquad \delta v' = -\delta w \sin\alpha + \delta v \cos\alpha \quad (19)$$

where δ is a small deviation from equilibrium. The angle α is determined by the following relation:

$$\alpha = \frac{1}{2}\tan^{-1}\left[\frac{(1-2w)p^{1/2}}{w(1-w)+p}\right] \quad (20)$$

The quantity Q in (18) is always positive because it contains the squares of the coordinates. From (17) it is seen that the condition for both frequencies to be real is

$$\frac{1}{1-w} \geq \frac{3w}{Q} \quad (21)$$

For the frequency ω_- to be real, (17) requires $Q \geq 3w(1-w)$. For any $w < 0$ (which is the allowed range of w), this inequality is satisfied: the left-hand side is always positive while the right-hand side is always negative.

For the frequency ω_+ to be real, the following function, $F(w)$, must be positive (in accordance with (17) and (18))

$$F(w) = (w^2 + p)[(1-w)^2 + p] - 9w^2(1-w)^2 \quad (22)$$

After replacing w with $\gamma = (1 - 1/w)^{1/3}$, (22) becomes

$$F(\gamma) = \gamma^2(\gamma^2 - 1)^2(1 + 4\gamma^2 + \gamma^4)(\gamma^3 - 1)^{-4} \quad (23)$$

Because the allowed range of $w < 0$ corresponds to $\gamma > 1$, it is seen that $F(\gamma)$ is always positive.

Thus, the corresponding classical energy terms correspond to the stable motion.

3. Electronic motion and the validity of the scenario

Now we unfreeze the slow subsystem and analyse a slow revolution of the axis connecting the proton and electron, the electron executing a circular orbit. In accordance with the concept of separating rapid and slow subsystems, the rapid subsystem (the revolving muon) follows the adiabatic evolution of the slow subsystem. This means that the slow subsystem can be treated as a modified "rigid rotator" consisting of the electron, the proton, and the ring, over which the muon charge is uniformly distributed, all distances within the system being fixed (see Fig. 1).

The potential energy of the electron in atomic units (with the angular momentum term) is

$$E_e = \frac{M^2}{2R^2} - \frac{1}{R} + [\rho^2 + (R - z)^2]^{-1/2} \quad (24)$$

where M is the electronic angular momentum. Its derivative by R must vanish at equilibrium, which yields

$$\frac{dE_e}{dR} = -\frac{M^2}{R^3} + \frac{1}{R^2} - (R - z)[\rho^2 + (R - z)^2]^{-3/2} = 0 \quad (25)$$

which gives us the value of the scaled angular momentum

$$\ell_e = \frac{M}{R^{1/2}} \quad (26)$$

corresponding to the equilibrium

$$\ell_e^2 = 1 - (1-w)[(1-w)^2 + p]^{-3/2} \quad (27)$$

where the scaled quantities w and p of the muon coordinates are defined in (5). Using the muon equilibrium condition from (7) with v^2 denoted p, we can represent (27) in the form

$$\ell_e^2 = 1 + w(w^2 + p)^{-3/2} \quad (28)$$

After replacing w with $\gamma = (1 - 1/w)^{1/3}$, we obtain

$$\ell_e(\gamma) = [1 - (1-\gamma)^2(1 + \gamma + \gamma^2)^{1/2}(1 - \gamma + \gamma^2)^{-3/2}]^{1/2} \quad (29)$$

The electron revolution frequency is $\omega = M/R^2 = \ell_e(\gamma)/R^{3/2}$ given that $M = \ell_e(\gamma)R^{1/2}$ in accordance with (26). Because $R = L^2 r(\gamma)/m$ (see (5)) with $r(\gamma)$ given by (12), then from $\omega = \ell_e(\gamma)/R^{3/2}$ we obtain

$$\omega = \frac{m^{3/2}\ell_e(\gamma)}{L^3[r(\gamma)]^{3/2}} \quad (30)$$

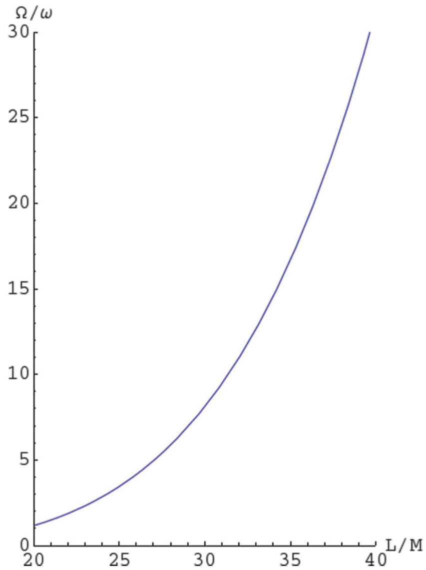
From (15) and (30) we find the following ratio of the muon and electron revolution frequencies:

$$\frac{\Omega}{\omega} = \frac{m^{-1/2}f(\gamma)[r(\gamma)]^{3/2}}{\ell_e(\gamma)} \quad (31)$$

where $f(\gamma)$ is given in (15).

In addition to the preceding relation, $R = L^2 r(\gamma)/m$, the same quantity R can be expressed from (26) as $R = M^2[\ell_e(\gamma)]^{-2}$. Equating the right-hand sides of these two expressions, we obtain the equality $L^2 r(\gamma)/m = M^2[\ell_e(\gamma)]^{-2}$, from which it follows

Fig. 4. The ratio of the muon and electron revolution frequencies, Ω/ω, versus the ratio of the muon and electron angular momenta, L/M.



$$\frac{L}{M} = \frac{m^{1/2}}{\ell_e(\gamma)[r(\gamma)]^{1/2}} \tag{32}$$

The combination of (31) and (32) represents an analytical dependence of the ratio of the muon and electron revolution frequencies Ω/ω versus the ratio of the muon and electron angular momenta L/M via the parameter γ as the latter varies from 1 to ∞. This dependence is presented in Fig. 4.

For the separation into the rapid and slow subsystems to be valid, the ratio of frequencies Ω/ω should be significantly greater than unity. From Fig. 4 it is seen that this requires the ratio of angular momenta L/M to be noticeably greater than 20.

There is another validity condition to be checked for this scenario. Namely, the revolution frequency, Ω, of the muon must also be much greater than the inverse lifetime of the muon $1/T_{\text{life}}$, where $T_{\text{life}} = 2.2\ \mu\text{s} = 0.91 \times 10^{11}$ a.u.: $\Omega T_{\text{life}} \gg 1$. Because for almost all values of R, the muon revolution frequency, Ω, is very close to its maximum value, $\Omega_{\max} = m/L^3$, as shown in Sect. 2, then the second validity condition can be estimated as $(m/L^3)T_{\text{life}} \gg 1$, from which it follows

$$L \ll L_{\max} = (mT_{\text{life}})^{1/3} = 26\,600 \tag{33}$$

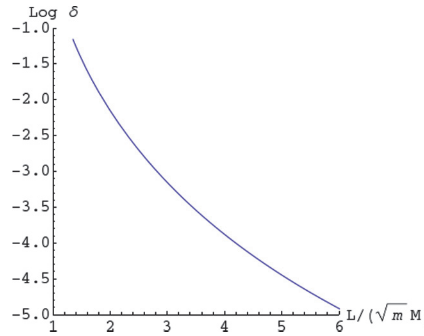
(recall that $m = 206.768\,274\,6$ in atomic units). So, the second validity condition is fulfilled for any practically feasible value of the muon angular momentum, L.

Thus, for the ratio of angular momenta L/M noticeably greater than 20, we deal here with a muonic quasi molecule where the muon rapidly rotates about the axis connecting the proton and electron following a relatively slow rotation of this axis.

4. Red shift of spectral lines compared to muonic hydrogen atoms

The muon, rotating in a circular orbit at frequency $\Omega(R)$, should emit a spectral line at this frequency. The maximum value $\Omega_{\max} = m/L^3$ corresponds to the frequency of spectral lines emitted by the muonic hydrogen atom (by the μp-subsystem). For the equilibrium value of the proton–electron separation — just as for almost all values of R — the frequency Ω is slightly smaller than $\Omega_{\max}$. Therefore, the spectral lines, emitted by the muon in the quasi-molecule μpe, experience a red shift compared to the corresponding spectral lines that would have been emitted by the muon in a muonic hydrogen atom. The relative red shift, δ, is defined as follows:

Fig. 5. Universal dependence of the relative red shift, δ, of the spectral lines of the quasimolecule μpe (or πpe) on $L/(m^{1/2}M)$, which is the ratio of the muon and electron angular momenta, L/M, divided by the square root of the mass, m, of the muon or pion.



$$\delta = \frac{\lambda - \lambda_0}{\lambda_0} = \frac{\Omega_{\max} - \Omega}{\Omega} \tag{34}$$

where λ and λ_0 are the wavelengths of the spectral lines for the quasi molecule μpe and the muonic hydrogen atom, respectively. Using (15), the relative red shift can be represented in the form

$$\delta(\gamma) = \frac{1}{f(\gamma)} - 1 \tag{35}$$

where $f(\gamma)$ is given in (15).

The combination of (35) and (32) represents an analytical dependence of the relative red shift δ on the ratio of the muon and electron angular momenta L/M via the parameter γ as the latter varies from 1 to ∞. Figure 5 presents the dependence of δ on $L/(m^{1/2}M)$. In this form the dependence is "universal" (i.e., valid for different values of the mass m), for example, it is valid also for the quasi molecule πpe where there is a pion instead of the muon. Figure 6 presents the dependence of δ on L/M specifically for the quasi molecule μpe.

It is seen that the relative red shift of the spectral lines is well within the spectral resolution $\Delta\lambda_{\text{res}}/\lambda$ of available spectrometers: $\Delta\lambda_{\text{res}}/\lambda \sim (10^{-4}$–$10^{-5})$ as long as the ratio of the muon and electron angular momenta, $L/M < 80$. Thus, this red shift can be observed and this would be one of the ways to detect the formation of such muonic negative hydrogen ions.

Figure 7 presents the dependence of the relative red shift δ on the ratio of the muon and electron revolution frequencies Ω/ω. It is seen that the relative red shift decreases as the ratio of the muon and electron revolution frequencies increases, but it remains well within the spectral resolution $\Delta\lambda_{\text{res}}/\lambda$ of available spectrometers.

5. Conclusion

We studied a muonic–electronic negative hydrogen ion. We demonstrated that in this case, the muonic motion can represent a rapid subsystem while the electronic motion can represent a slow subsystem — a result that might seem counterintuitive. In other words, the muon rapidly revolves in a circular orbit about the axis connecting the proton and electron while this axis slowly rotates following a relatively slow electronic motion.

Fig. 6. Dependence of the relative red shift, δ, of the spectral lines of the quasimolecule μpe on the ratio of the muon and electron angular momenta, *L*/*M*.

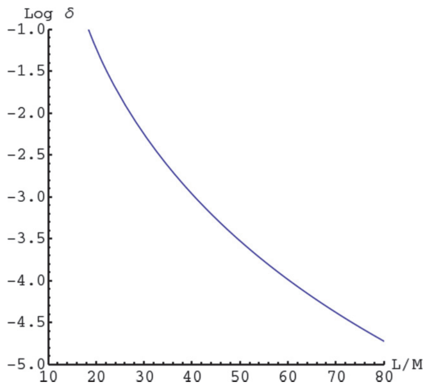
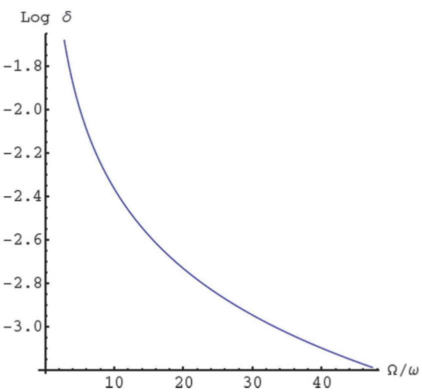


Fig. 7. Dependence of the relative red shift, δ, on the ratio of the muon and electron revolution frequencies, *Ω*/*ω*.



We used a classical analytical description to find the energy terms of such a system, (i.e., dependence of the energy of the muon on the distance between the proton and electron). We found that there is a double-degenerate energy term. We demonstrated that it corresponds to a stable motion. We also conducted an analytical relativistic treatment of the muonic motion, which is presented in Appendix A. It was found that the relativistic corrections are relatively small. Their relative value is $\sim 1/(cL)^2 \sim 0.5 \times 10^{-4}/L^2$ (we remind the readers that here $c = 137.036$ is the speed of light in atomic units).

Then we unfroze the slow subsystem and analysed a slow revolution of the axis connecting the proton and electron. The slow subsystem can be treated as a modified "rigid rotator" consisting of the electron, the proton, and the ring, over which the muon charge is uniformly distributed, all distances within the system being fixed. We derived the condition required for the validity of the separation into the rapid and slow subsystems.

Finally we showed that the spectral lines, emitted by the muon in the quasimolecule μpe, experience a red shift compared to the corresponding spectral lines that would have been emitted by the muon in a muonic hydrogen atom (in the μp-subsystem). The relative values of this red shift, which is a "molecular" effect, are significantly greater than the resolution of available spectrometers and thus can be observed. Observing this red shift should be one of the ways to detect the formation of such muonic–electronic negative hydrogen ions.

It should be noted that circular states of atomic and molecular systems is an important subject in its own right. They have been extensively studied both theoretically and experimentally for several reasons (see, e.g., refs. 14–18 and references therein): (*i*) they have long radiative lifetimes and highly anisotropic collision cross sections, thereby enabling experiments on inhibited spontaneous emission and cold Rydberg gases; (*ii*) these classical states correspond to quantal coherent states, objects of fundamental importance; (*iii*) a classical description of these states is the primary term in the quantal method based on the 1/*n*-expansion; and (*iv*) they can be used in developing atom chips. In the present paper we used circular states just to get the message across and to stimulate further studies of muonic–electronic negative hydrogen ions.

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Appendix A. Relativistic treatment of the muonic motion

The Hamiltonian of the rotating muon is

$$H = c\left(m^2c^2 + p_z^2 + p_\rho^2 + \frac{p_\varphi^2}{\rho^2}\right)^{1/2} - Z(z^2 + \rho^2)^{-1/2} - Z'[(R - z)^2 + \rho^2]^{-1/2} - mc^2 \tag{A1}$$

Because φ is a cyclic coordinate, the corresponding momentum is conserved

$$|p_\varphi| = \text{const.} = L \tag{A2}$$

With this substituted into (A1) and taking into account that in a circular state, $p_z = p_\rho = 0$, we obtain the energy of the muon in a circular state

$$E = c\left(m^2c^2 + \frac{L^2}{\rho^2}\right)^{1/2} - Z(z^2 + \rho^2)^{-1/2} - Z'[(R - z)^2 + \rho^2]^{-1/2} - mc^2 \tag{A3}$$

With $Z = 1$, $Z' = -1$, and the scaled quantities

$$w = \frac{z}{R} \qquad v = \frac{\rho}{R} \qquad \varepsilon = -ER \qquad \ell = \frac{L}{mcR} \qquad r = \frac{R}{L} \tag{A4}$$

we obtain the scaled energy, ε, of the muon

$$\varepsilon = (w^2 + v^2)^{-1/2} - [(1 - w)^2 + v^2]^{-1/2} + mc^2R\left[1 - \left(1 + \frac{\ell^2}{v^2}\right)^{1/2}\right] \tag{A5}$$

The equilibrium condition with respect to the scaled coordinate, w, is $\partial\varepsilon/\partial w = 0$, which yields

$$p(w) = w^{2/3}(w - 1)^{2/3}[w^{2/3} + (w - 1)^{2/3}] \tag{A6}$$

where $p = v^2$.

The equilibrium condition with respect to the scaled coordinate v is $\partial\varepsilon/\partial v = 0$, which yields

$$\ell^2 = \left(1 + \frac{\ell^2}{p}\right)^{1/2}\{(w^2 + p)^{-3/2} - [(1 - w)^2 + p]^{-3/2}\}\frac{p^2}{mc^2R} \tag{A7}$$

From the relation before last in (A4), we find $R = L/(mc\ell)$. Substituting this into (A7), we can solve it for ℓ and obtain

$$\ell = \left(\left(\frac{c^2L^2}{p^4}\right)\{(w^2 + p)^{-3/2} - [(1 - w)^2 + p]^{-3/2}\}^{-2} - \frac{1}{p}\right)^{-1/2} \tag{A8}$$

From the last two relations in (A4), we find $r = 1/(mc\ell)$; thus

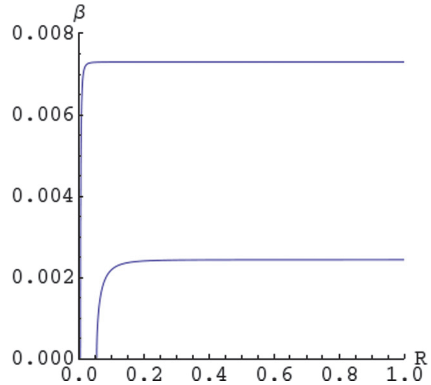
$$r = \left\{\left[\frac{cL}{p^2 g(w, p)}\right]^2 - \frac{1}{p}\right\}^{1/2}(mc)^{-1}$$
$$g(w, p) = (w^2 + p)^{-3/2} - [(1 - w)^2 + p]^{-3/2} \tag{A9}$$

where p is given by (A6). Therefore, the quantity r in (A9) is the scaled "internuclear" distance dependent on the scaled internuclear coordinate, w, for a given absolute value of the angular momentum projection on the internuclear axis, L.

Now we substitute $R = L/(mc\ell)$ and the value of ℓ from (A8), as well as the value of p from (A6) into (A5), obtaining $\varepsilon(w, L)$ — the scaled energy of the muon dependent on the scaled internuclear coordinate, w, for a given value of the angular momentum, L. Because $E = -\varepsilon/R$ and $R = rL$, $E = -\varepsilon_1/L$ where $\varepsilon_1 = \varepsilon/r$. The parametric dependence $E(R)$, where $E = -\varepsilon_1/L$ and $R = Lr$, will yield the energy terms for a given value of L.

After introducing the parameter $\gamma = (1 - 1/w)^{1/3}$, the parametric dependence takes the following form:

Fig. A1. The ratio, β, of the muon velocity to the speed of light versus the "internuclear" distance, R (a.u.), for $L = 1$ (the upper curve) and $L = 3$ (the lower curve).



$$E(\gamma, L) = -mc^2\left\{1 + (\sigma^6 - \tau)^{-1/2}\left[\frac{\tau}{\sigma(1 + \gamma + \gamma^2)} - \sigma^3\right]\right\} \tag{A10}$$

$$R(\gamma, L) = \frac{(L^2/m)(\sigma^6 - \tau)^{1/2}}{\gamma(1 + \gamma^2)^{1/2}} \tag{A11}$$

where quantities σ and τ are defined as follows:

$$\sigma = \left(\frac{1 + \gamma^2 + \gamma^4}{1 + \gamma^2}\right)^{1/2} \qquad \tau = \left(\frac{1 - \gamma^3}{cL}\right)^2 \tag{A12}$$

The revolution frequency of the muon is

$$\omega = \frac{mc^2}{L}\left(1 - \frac{\tau}{\sigma^6 - \tau}\right)^{1/2}\frac{\tau}{\sigma^6 - \tau} \tag{A13}$$

Let us check the degree of the relativity of the muon motion. Figure A1 shows the ratio β of the muon velocity to the speed of light versus the "internuclear" distance, R, for $L = 1$ and 3. It is seen that for all values of $R \approx n^2$ ($n = 1, 2, 3$), this ratio is practically equal to some constant value, β_{max}. It is easy to find that $\beta_{max} = 1/(cL) = 1/(137.036L)$, because $c = 137.036$ in a.u.

It is interesting to compare the preceding β_{max} with the corresponding average value of β_e for the electron motion in hydrogen atoms: $\beta_e = 1/(cn)$. So, β_{max} for the muon motion differs from β_e for the electron motion in hydrogen atoms only by the substitution of the principal quantum number, n, of the electron by the angular momentum quantum number, L, of the muon.

Thus, even for $L = 1$ (for which β_{max} is the highest), the muon motion is only weakly relativistic. The relativistic correction to the average frequency of the muon radiation is $\sim 1/(cL)^2$ (a.u.), where $c = 137.036$. Thus, the relative correction is insignificant even for $L \sim 1$ and it rapidly diminishes as L grows: for example, it is $\sim 10^{-5}$ for $L = 3$ and $\sim 10^{-7}$ for $L = 15$.