Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 12, 2011
1:00PM to 3:00PM
Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. The Dirac equation can be written in terms of two Pauli-type 2-component spinors, ϕ and χ. For a free particle of mass m and momentum $\vec{p}$, these are plane waves whose spinor coefficients satisfy

$$m\phi + \vec{p}\cdot\vec{\sigma}\chi = E\phi \,, \tag{1}$$

$$\vec{p}\cdot\vec{\sigma}\phi - m\chi = E\chi \,, \tag{2}$$

where E is the energy and $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices. We work in natural units $c = \hbar = 1$.

(a) Solve Eqs. (1) and (2) for E and give a physical interpretation of every solution for the given $\vec{p}$.

(b) The scattering by a spin-independent central potential, V, between the initial and final momenta $\vec{p}$ and $\vec{p}\,'$, is described in the Born approximation by the matrix element

$$V(\vec{p}\,',\vec{p})\left[\phi^\dagger_{\vec{p}\,'}\phi_p + \chi^\dagger_{\vec{p}\,'}\chi_p\right] \,, \tag{3}$$

where $V(\vec{p}\,',\vec{p})$ is the matrix element of V between momentum eigenstates. Using your solution to part (a), show that

$$\chi^\dagger_{\vec{p}\,'}\chi_p = \phi^\dagger_{\vec{p}\,'}\Sigma(\vec{p}\,',\vec{p})\,\phi_p \,, \tag{4}$$

where $\Sigma(\vec{p}\,',\vec{p}) = F_1(\vec{p}\,',\vec{p}) + \vec{\sigma}\cdot\vec{F}_1(\vec{p}\,',\vec{p})$. Determine F_1 and F_2.

(c) Suppose that the electron is initially moving in the x_3-direction, with its spin aligned in the x_3-direction, and that it is scattered by a very weak potential, V, into the x_2-direction. Show that, in the non-relativistic limit, the probability of *spin flip* in the course of such a scattering event is small. Quantify what is meant by "small" here.

2. Two particles with masses m_1 and m_2 and coordinates x_1 and x_2 move on a circle of radius R, $0 \leq x_1,\, x_2 < 2\pi R$.

   (a) If the particles are non-interacting find the allowed energies and corresponding energy eigenfunctions.

   (b) Next a short range potential $V(|x_1 - x_2|)$ is introduced. The effects of the potential V are described by a vanishing reflection coefficient R and a transmission coefficient $T = e^{i2\delta}$. Find the new energy eigenvalues and eigenfunctions.

3. Time-dependent and time-independent perturbation in a 1D Harmonic Oscillator.

   Consider an isotropic one-dimensional harmonic oscillator where the unperturbed Hamiltonian is given by:

$$\tilde{H}_0 = \frac{p^2}{2m} + \frac{mw_0^2}{2}x^2$$

   We now consider the effect of the perturbation Hamiltonian

$$\tilde{H}' = \lambda x f(t)$$

   where λ is a constant, and $f(t)$ is a time-dependent dimensionless function.

   (a) Assume that the perturbation Hamiltonian is time-independent, i.e. $f(t) = 1$ for all t. Find the energy shifts of the ground state to the first non-zero order in λ.

   (b) Now, assume that the perturbation is time-dependent, such that

$$f(t) = \int_{-\infty}^{+\infty} \rho(w) \{e^{iwt} + e^{-iwt}\} dw$$

   where $\rho = \sqrt{\frac{1}{\pi w_0^2}} e^{-(w-w_0)^2/w_0^2}$ with the constant w_0 given in the harmonic oscillator. Assume that the oscillator is in its ground state at $t = 0$. What are the excited states of this oscillator that the ground state can make a transition into?

   (c) Find the transition rate into these excited states using Fermi's golden rule.

4. The wave function for the first excited state of the simple harmonic oscillator is:

$$U_1(x) = \left(\frac{m\omega}{\hbar}\right)^{3/4} \sqrt{\frac{2}{\sqrt{\pi}}} x e^{-\frac{m\omega x^2}{2\hbar}}$$

with $E_1 = \frac{3}{2}\hbar\omega$. (The Hamiltonian is $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$).

Suppose the wave function of a particle in the oscillator is:

$$U(x) = \begin{cases} cU_1(x) & x < 0; \\ 0 & x > 0. \end{cases}$$

at $t = 0$.

i. What is c?

ii. What is the expectation value of the energy, $\langle H \rangle$, at $t = 0$?

iii. What is the probability P_1 that the particle be found in the first excited state at $t = 0$?

iv. How does P_1 change with time?

5. A particle is placed in a symmetric potential well of depth U and width a (such that the well is in the range $[-\frac{a}{2} \ldots \frac{a}{2}]$). The particle has only one bound state, with the binding energy $\epsilon = U/2$. calculate the probabilities of finding the particle in the classically-allowed and classically-forbidden regions.

   Note: The following integrals *may* be helpful:

$$\int \sin^2(kx)\,dx = \frac{x}{2} - \frac{1}{4k}\sin(2kx) + C; \tag{5}$$

$$\int \cos^2(kx)\,dx = \frac{x}{2} + \frac{1}{4k}\sin(2kx) + C. \tag{6}$$

Quantum mechanics

a) Given : ① $m\phi + \vec{p}\cdot\vec{\sigma}\,\chi = E\phi$

② $\vec{p}\cdot\vec{\sigma}\,\phi - m\chi = E\chi$

From ② $$\boxed{\chi = \frac{1}{E+m}\,\vec{p}\cdot\vec{\sigma}\,\phi} \quad (*)$$

Replace in ① : $(\vec{p}\cdot\vec{\sigma})(\vec{p}\cdot\vec{\sigma})\,\phi = \vec{p}^{\,2}\phi = (E+m)(E-m)\phi$

$$\Rightarrow \left[E^2 - (m^2 + \vec{p}^{\,2})\right]\phi = 0$$

which is solved by $E = \pm\sqrt{m^2 + \vec{p}^{\,2}}$.

These are the positive and negative energy solution found by Dirac:

- $E > 0$: describes "particles"
- $E < 0$: describes "anti-particles"

b) From (*)

$$\chi^\dagger_{\vec{p}}\,\chi_{\vec{p}'} = \frac{1}{(E+m)(E'+m)}\,\phi^\dagger_{\vec{p}}(\vec{p}\cdot\vec{\sigma})(\vec{p}'\cdot\vec{\sigma})\,\phi_{\vec{p}'}$$

Using $$p^i p'^j \sigma^i \sigma^j = p^i p'^j\left[\delta^{ij} + i\epsilon^{ijk}\sigma^k\right]$$
$$= \vec{p}\cdot\vec{p}' + i(\vec{p}\times\vec{p}')\cdot\vec{\sigma}$$

we see that

$$\chi^\dagger_{\vec{p}}\,\chi_{\vec{p}'} = F_1(\vec{p},\vec{p}') + \vec{\sigma}\cdot\vec{F}_2(\vec{p},\vec{p}')$$

$$F_1(\vec{p},\vec{p}') = \frac{\vec{p}\cdot\vec{p}'}{(E+m)(E'+m)}$$

$$F_2(\vec{p},\vec{p}') = \frac{i\,\vec{p}\times\vec{p}'}{(E+m)(E'+m)}$$

c) For $\vec{p} = (0,0,p)$, $\vec{p}' = (0,p',0)$ and $\phi_{\vec{p}} = \begin{pmatrix}1\\0\end{pmatrix}$

so that $\hat{p}\cdot\vec{\sigma}\,\phi = +\phi$, we want the amplitude for

finding $\phi_{\vec{p}'} = \begin{pmatrix}0\\1\end{pmatrix}$:

$$\phi^{\dagger}_{\vec{p}'}\phi_{\vec{p}} = 0$$

$$\chi^{\dagger}_{\vec{p}'}\chi_{\vec{p}} = \phi^{\dagger}_{\vec{p}'} F_2(\vec{p}',\vec{p})\,\vec{\sigma}\,\phi_{\vec{p}}$$

$$= \frac{+ip'p}{(E+m)(E'+m)}\underbrace{\phi^{\dagger}_{\vec{p}'}\sigma^1\phi_{\vec{p}}}_{(0,1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1}$$

For low-velocity and elastic scattering:

$$E = E' \approx m + \frac{p^2}{2m} \approx m$$

$$\Rightarrow \quad \chi^{\dagger}_{\vec{p}'}\chi_{\vec{p}} \approx i\frac{p^2}{4m^2} = \frac{i}{2m}\overbrace{KE}^{\text{kinetic energy}}$$

so that $\text{probability} \propto \frac{(KE)^2}{4m^2} \ll 1$

Suggested Solution

Sec 3 - # 2

1. (a) For two independent particles we use the product of the single particle wave functions with energy that is the sum of the single particle energies:

$$\psi(x_1, x_2) = \frac{1}{2\pi R} e^{ip_{n_1}x_1} e^{ip_{n_2}x_2} \tag{1}$$

$$E_{n_1,n_2} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \tag{2}$$

where $p_1 = n_i/R$ for $i = 1, 2$ and n_1 and n_2 are integers.

(b) When interactions are introduced such a two-particle probem can be simplified using relative (x) and center of mass (X) coordinates:

$$x = x_2 - x_1 \tag{3}$$

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{4}$$

in terms of which the Hamiltonian becomes:

$$H = \frac{P^2}{2(m_1 + m_2)} + \frac{p^2}{2m_r} + V(|x|) \tag{5}$$

where P and p are conjugate to X and x respectively and the reduced mass $m_r = m_1 m_2/(m_1 + m_2)$. For infinite R, a complete set of energy and total momentum eigenstates can be written:

$$\psi(x, X) = e^{iXP} \left\{ \begin{array}{ll} e^{ipx} & x < 0 \\ e^{2i\delta} e^{ipx} & x > 0 \end{array} \right\}. \tag{6}$$

(Note: Here p can be postive or negative.)

For finite R we must require that the phase of the above wave function changes by 2π times an integer when either x_1 or x_2 is increased by $2\pi R$:

$$2\pi R(\frac{m_1}{m_1+m_2}P+p)+2\delta = 2\pi n_1 \tag{7}$$

$$2\pi R(\frac{m_2}{m_1+m_2}P-p)-2\delta = 2\pi n_2 \tag{8}$$

$$\tag{9}$$

which gives the altered quantization conditions:

$$P = \frac{n_1+n_2}{R} \tag{10}$$

$$\frac{m_1-m_2}{m_1+m_2}P+2p = \frac{2\pi(n_1-n_2)-4\delta}{2\pi R} \tag{11}$$

or

$$P = \frac{n_1+n_2}{R} \tag{12}$$

$$p = \frac{\pi(n_1-n_2)-2\delta}{2\pi R}-\frac{m_1-m_2}{m_1+m_2}\cdot\frac{n_1+n_2}{2R} \tag{13}$$

Quantum

Using lowering & raising operator $a, a^\dagger$ $\left(\begin{array}{l} a|n\rangle = \sqrt{n}|n-1\rangle \\ a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \end{array} \right)$

Then $\hat{H}_0 = \hbar\omega_0(a^\dagger a + \frac{1}{2})$, $\hat{H}' = \lambda f(t)\sqrt{\frac{\hbar}{2m\omega_0}}(a + a^\dagger)$

(a) $f(t) = 1$ $\langle n|\hat{H}'|n\rangle = 0 \rightarrow$ requires the second order in λ for non-zero perturbation term

Also note that

$$\langle n|\hat{H}'|0\rangle = \lambda\sqrt{\frac{\hbar}{2m\omega_0}}\,\delta_{n,1}$$

$$\langle n|\hat{H}'|1\rangle = \lambda\sqrt{\frac{\hbar}{2m\omega_0}}\,(\delta_{n,0} + \sqrt{2}\,\delta_{n,2})$$

Then for the ground state energy shift

$$\Delta E_0 \simeq \sum_{n=1}^{\infty} \frac{|\langle n|\hat{H}'|0\rangle|^2}{\hbar\omega_0(\frac{1}{2}) - \hbar\omega_0(n+\frac{1}{2})} = -\frac{\lambda^2}{\hbar\omega_0}\left(\frac{\hbar}{2m\omega_0}\right) = -\frac{\lambda^2}{2m\omega_0^2}.$$

Likewise for the first excited state,

$$\Delta E_1 \simeq \sum_{n\neq 1} \frac{|\langle n|\hat{H}'|0\rangle|^2}{\hbar\omega_0(\frac{3}{2}) - \hbar\omega_0(n+\frac{1}{2})} = \lambda^2\left(\frac{\hbar}{2m\omega_0}\right)\left[\frac{1}{\hbar\omega_0} + \frac{\sqrt{2}^2}{-\hbar\omega_0}\right]$$

$$= -\frac{\lambda^2}{2m\omega_0^2}.$$

(b) $|\psi(t)\rangle = \sum_n C_n(t) e^{-iE_n} |n\rangle$

where

$$C_n(t) \simeq C_n(0) + \frac{-i}{\hbar}\sum_{k\neq n}\int_0^t \langle n|\hat{H}'(t')|k\rangle \, C_k(0)\, e^{-i(E_k - E_n)t'/\hbar}\, dt'$$

from the time-dep perturbation theory

Since the initial condition states $C_k(0)=0$ for $k\neq 0$,

for $n \neq 0$

$$C_n(t) \propto \langle n|\tilde{H}'|0\rangle \propto \delta_{n,1}.$$

Thus only $0\to 1$ transition occurs.

(c) $$\tilde{H}' = \int_{-\infty}^{\infty} d\omega\, P(\omega)\left[\tilde{V} e^{i\omega t} + \tilde{V} e^{-i\omega t}\right]$$

where $$\tilde{V} = \lambda\sqrt{\frac{\hbar}{2m\omega_0}}\,(a+a^\dagger)$$

From Fermi-Golden rule,

$$\Gamma_{0\to 1}(\omega) = \frac{2\pi}{\hbar}\,|\langle 0|\tilde{V}|1\rangle|^2 \{\delta(E_1-E_0-\hbar\omega) + \delta(E_1-E_0+\hbar\omega)\}$$

Total transition rate is then

$$\Gamma_{0\to 1}^{tot} = \int_{-\infty}^{\infty} d\omega\, P(\omega)\, \Gamma_{0\to 1}(\omega)$$

$$= \frac{2\pi}{\hbar^2}\left(P(\omega_0) + P(-\omega_0)\right)|\langle 0|\tilde{V}|1\rangle|^2$$

$$= \frac{2\pi}{\hbar^2}\times 2\cdot\sqrt{\frac{1}{\pi\omega_0^2}}\cdot\lambda^2\frac{\hbar}{2m\omega_0} = \underline{\frac{\sqrt{\pi}\,\lambda^2}{m\hbar\omega_0^2}}$$

Answer: (i) $1 = c^2 \int_{-\infty}^{0} dx\, |u_1(x)|^2 = \frac{c^2}{2} \int_{-\infty}^{\infty} |u_1(x)|^2 dx = c^2/2 \Rightarrow c = \sqrt{2}$

(ii) $\langle H \rangle = \int_{-\infty}^{\infty} dx\, u(x)\, H\, u(x) = 2 \int_{-\infty}^{0} dx\, u(x)\, \underbrace{H\, u(x)}_{E_1 u_1(x)}$

$$= 2 E_1 \underbrace{\int_{-\infty}^{0} dx\, |u_1(x)|^2}_{1/2} = E_1 .$$

(iii) $P_1(t=0) = \left| \int_{-\infty}^{\infty} dx\, u_1(x)\, u(x) \right|^2 = \left| c \int_{-\infty}^{0} dx\, |u_1(x)|^2 \right|^2 = |c/2|^2 = \frac{1}{2}$

(iv) $P_1(t) = P_1(t=0)$; no change in time.

MODERN PHYSICS – QUANTUM MECHANICS

Life in the forbidden region. SOLUTION.

In both forbidden regions the wave function decays as $\exp(-\kappa|x|)$, where $\hbar^2\kappa^2/2m = \epsilon$. In the allowed region, the wave function oscillates as $\cos kx$, where $\hbar^2 k^2/2m = U - \epsilon$. Since $\epsilon = U/2$,

$$\kappa = k = \sqrt{\frac{mU}{\hbar^2}}. \tag{1}$$

With the symmetric x-coordinate choice, the matching of the wave function and its derivative on the edge of the well (for example at $x = a/2$) gives

$$A\cos\left(\frac{ka}{2}\right) = Be^{-ka/2}, \quad -Ak\sin\left(\frac{ka}{2}\right) = -kBe^{-ka/2}. \tag{2}$$

The ratio of these equations yields

$$\tan\left(\frac{ka}{2}\right) = 1 \quad \rightarrow \quad ka = \frac{\pi}{2}. \tag{3}$$

The probability for the particle to exist in the forbidden region is

$$P_{\text{out}} = 2\int_{a/2}^{\infty} dx\, B^2 e^{-2kx} = \frac{B^2}{k}e^{-ka}. \tag{4}$$

From matching conditions (2, 3) this is equivalent to

$$P_{\text{out}} = \frac{A^2}{k}\cos^2\left(\frac{ka}{2}\right) = \frac{A^2}{2k}. \tag{5}$$

Analogously, the probability to exist in the allowed region is

$$P_{\text{in}} = 2\int_0^{a/2} dx\, A^2\cos^2(kx) = \frac{A^2}{2k}[ka + \sin(ka)] = \frac{A^2}{2k}\left(\frac{\pi}{2} + 1\right). \tag{6}$$

The ratio of the two probabilities is

$$\frac{P_{\text{out}}}{P_{\text{in}}} = \frac{2}{2+\pi}. \tag{7}$$

Since $P_{\text{out}} + P_{\text{in}} = 1$, we obtain

$$P_{\text{out}} = \frac{2}{4+\pi} \simeq 0.28, \quad P_{\text{in}} = \frac{2+\pi}{4+\pi} \simeq 0.72. \tag{8}$$