Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 10, 2011
3:10PM to 5:10PM
Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Classical model of diamagnetic susceptibility, $\tilde{\mathbf{M}} = \chi_m \tilde{\mathbf{B}}$ of permeable matter. Here $1 + \chi_m = \mu/\mu_0$ is the relative permeability. Consider a uniform magnetic induction $\tilde{\mathbf{B}} = B\hat{\mathbf{z}}$ and a simple Bohr model of neutral atoms of charge $Z|e|$. In the absence of a B field, the radius, r_0, of circular orbits is related to orbital velocity, v_0, through the balance of Coulomb and centripetal forces. With a small uniform B , the Lorentz force slightly modifies the velocity $v = v_0 + \delta v(B)$ of each electron to balance all forces. Assume r_0 remains fixed. (Neglect spin effects, orbital precession, and other complications here.)

   (a) Calculate $\delta v(B)$ to lowest linear order in the field strength B. From this estimate the average change in the orbital current, $\delta I(B)$, in the (xy) plane. Then compute the change in the atomic orbital magnetic dipole moment, $\delta\tilde{\mathbf{m}}$ induced to first order B.

   (b) Assume the number density of atoms per unit volume is $\rho = N/V$. What is the diamagnetic susceptibility, χ_m, in this model expressed in terms of $Z, |e|, r_0, m_e$, and ρ? Estimate the magnitude of χ_m assuming $\rho = 1/(4\pi r_0^3/3)$. Use the hydrogenic Bohr radius for r_0 for a rough numerical estimate for water taking effective $Z \sim 3$.

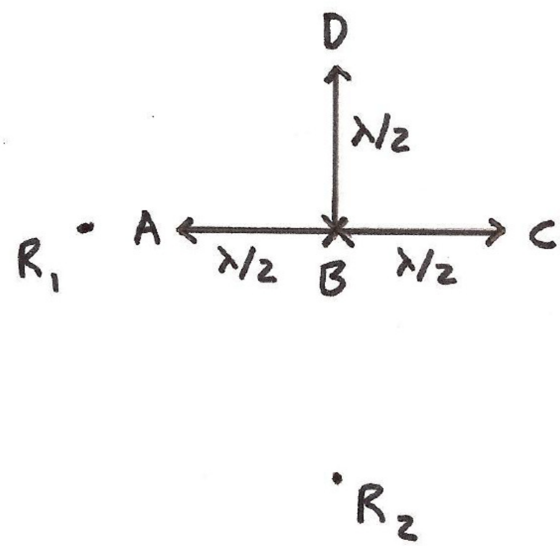
2. Four identical coherent monochromatic wave sources A, B, C, D as shown below produce waves of the same wavelength λ. Two receivers R_1 and R_2 are at great (but equal) distances from B.

   (a) What is the approximate relative signal size picked up by the two receivers?

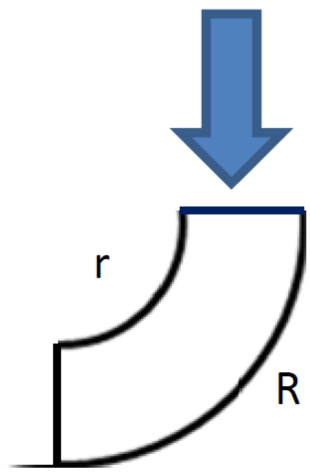
   (b) What is the approximate relative signal size picked up by the two receivers if source B is turned off?

   (c) What is the approximate relative signal size picked up by the two receivers if source D is turned off?

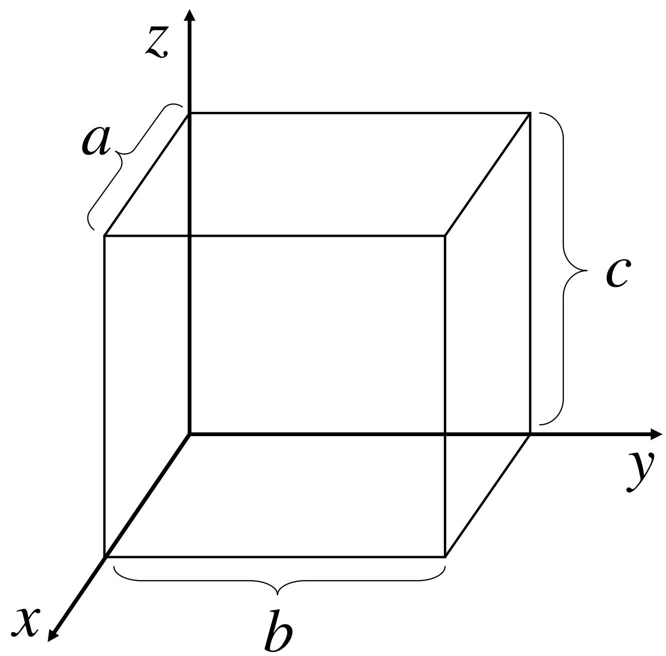
   (d) Which receiver can tell which source, B or D, has been turned off? Explain.



3. Consider a thick sheet of glass (index of refraction n) that is bent at one end so the cross section is as shown in the figure. The bend is perfectly cylindrical and the inner radius is r and the outer radius is R. A laser is pointing vertically down and enters the glass through the top side. What is the constraint that ensures that the laser light exists through the left (vertical) face and nowhere else?



4. Consider a box with side lengths a, b, and c along the x, y, and z axes. Suppose there is no electric charge inside the box and that $\phi = 0$ on the surface of the box except at $z = 0$ where $\phi = V_1 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)$, and at $z = c$ where $\phi = V_2 \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{2\pi y}{b}\right)$. Find ϕ everywhere inside the box.



5. The electric charge Q is uniformly distributed along the rod of length L.

   (a) Find the electric field at the distances from the rod:

      i. much larger than L.

      ii. much smaller than L and near the middle of the rod.

   (b) The same rod is now bent into a semicircular shape. Find the electric field in the center of the semicircle.

Solution part a:
For $B = 0$, $v_0^2 = Ze^2/(mr_0)$ balances the forces. For small B, $v = v_0 + \delta v$. The extra Lorentz force is $-|e|Bv/c\hat{\rho} \approx -|e|Bv_0/c\hat{\rho}$ to first order in B, where we assume for that the orbital angular momentum (projection) points parallel to $\vec{B} = B\hat{z}$ and that we can neglect precession and spin effects. Here $(\hat{\rho}, \hat{\phi}, \hat{\mathbf{z}})$ are cylindrical coordinate unit vectors.

Because of the small velocity shift, the centripetal force changes by $(m_e 2v_0\delta v/r_0)\hat{\rho}$. In order to balance all forces including the Lorentz, the velocity shift $\delta\vec{v} = |e|Br_0/(2mc)\hat{\phi}$ is independent of initial $\vec{v}_0$ and directed to resist the increase of magnetic flux through the atom (Lenz's law). The effective average orbital current induced around each atom is thus approximately

$$\delta\vec{\mathbf{I}} = Z|e|\delta v/(2\pi r_0)\hat{\phi} = (Ze^2/(4\pi mc))B\hat{\phi}$$

to first order in small B. Finally, the induced magnetic dipole moment of the atom treated as a current loop with area πr_0^2 is

$$\delta\vec{\mathbf{m}} = -\hat{\mathbf{z}}(\delta\vec{I} \cdot \hat{\phi})(\pi r0^2) = -(Ze^2 r_0^2/(4mc))\vec{\mathbf{B}}$$

Solution part b: The induced Magnetization $M = \delta m\rho = \chi_m B$ is just the net induced magnetic dipole of the atom times the atomic number density. Therefore

$$\chi_m = -(Ze^2 r_0^2 \rho/(4m_e c))$$

If we assume that the matter like water can be crudely approximated by close packed atoms with $\rho = 1/(4\pi r_0^3/3)$ and if we further assume a typical Bohr atomic radius $r_0 \sim \hbar/(\alpha m_e c)$ we get the simple estimate

$$\chi_m = Z_{eff}\alpha^2/(3\pi)$$

which for H_2O water $Z_{eff} = 8/3 + 2/3 \approx 3$ gives $\chi_m \sim -2 \times 10^{-5}$, which is within a factor 2 or so of the data.

Let r be the distance of R_1 and R_2 to B. Given $r \gg \lambda$.

Let amplitude of waves be E_0

At R_1: $E_{10} = E_0 e^{ik(r-\lambda/2)} + E_0 e^{ikr}$

$+ E_0 e^{ik(r+\lambda/2)} + E_0 e^{ik\sqrt{r^2+\lambda^2/4}}$

At R_2: $E_{20} = E_0 e^{ikr} + E_0 e^{ik(r+\lambda/2)}$

$+ 2E_0 e^{ik\sqrt{r^2+\lambda^2/4}}$

But $k\lambda = 2\pi \Rightarrow e^{ik\lambda/2} = -1$

For $r \gg \lambda$, $\sqrt{r^2+\lambda^2/4} \approx r \Rightarrow e^{ik\sqrt{r^2+\lambda^2/4}} \to e^{ikr}$

Plugging in

$E_{10} \approx 0$ and $E_{20} \approx 2E_0 e^{ikr}$

Intensity $I \propto |E|^2$

$I_1 = 0$ and $I_2 = 4E_0^2$

R_2 picks up greater signal.

(b) If B is turned off

$E_1 \approx -E_0 e^{ikr}$ $E_2 \approx E_0 e^{ikr}$

$I_1 = I_2 \sim E_0^2$

Two receivers pick up same intensity

(c) If source D is turned off

$E_1 \approx -E_0 e^{ikr}$ $E_2 \approx 3E_0 e^{ikr}$

$I_1 \sim E_0^2$ $I_2 \sim 9E_0^2$

R_2 picks up nine times greater signal.

(d) R_2 is the only receiver sensitive to the B and D source conditions.

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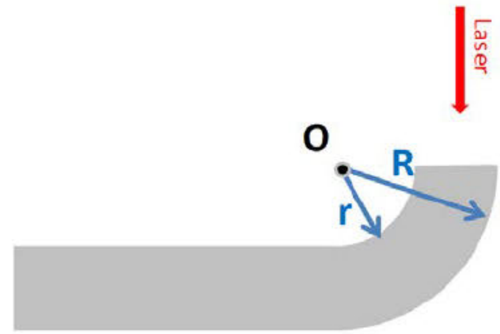
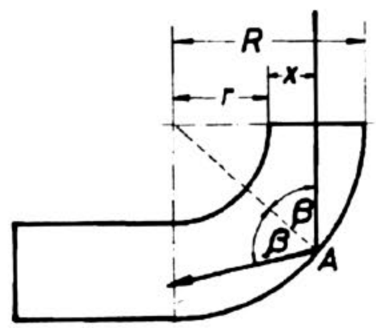


3. Consider a thick sheet of glass (index of refraction n) that is bent at one end so the cross section is as shown in the figure. The bend is perfectly cylindrical and the inner radius is r and the outer radius is R. A laser is pointing vertically down and enters the glass through the top side. What is the constraint that ensures that the laser light exists through the left (vertical) face and nowhere else?

Solution Example:

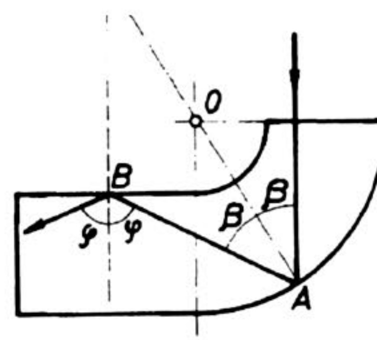


If β is larger than the angle of total internal reflection at point A, then the beam will stay inside the glass plate, consequently we have the condition:

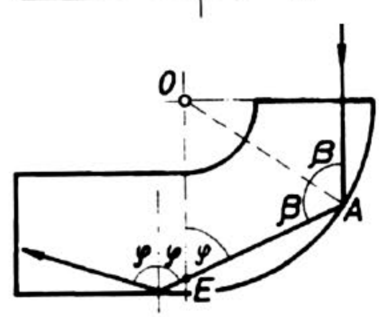
$$\sin\beta = \frac{x+r}{R} \geqslant \frac{1}{n}$$

$$\frac{r}{R} \geqslant \frac{1}{n} \quad \Rightarrow \quad \frac{R}{r} \leqslant n$$

The β angle is the is minimized when x = 0, providing the critical condition for the laser beam to stay inside the glass. Thus, the ratio of the larger radius to the smaller radius must remain below the index of refraction of the glass plate.



If the plate continues horizontally, as shown in the figures in the right, there can be one or more additional internal reflections. In some case the first reflection will happen on the lower side, in others the upper side of the horizontal portion. In either case the geometry ensures that φ is always larger than β, therefore the condition of total internal reflection is automatically satisfied.



Consequently, the $\frac{R}{r} \leqslant n$ condition is always sufficient to ensure good quality for a light guide (e.g., for a radio where the front panel is far from the laser diode used as indicator on the PCB). LEDs have broader angular emission than lasers, therefore the question is slightly more complicated for LEDs. However the result is still a decent rule of thumb for nice laboratory design.

Answer: In general $\nabla^2\phi = 0$ in the box gives

$$\phi(x,y,z) = \sum_{m_1,m_2} \sin\frac{m_1\pi x}{a}\sin\frac{m_2\pi y}{b}\left(A_{m_1m_2}\sinh K_{m_1m_2}z + B_{m_1m_2}\cosh K_{m_1m_2}z\right)$$

because $\phi|_{x=0,a} = \phi|_{y=0,b} = 0$. In the above

$$K^2_{m_1m_2} = \left(\frac{m_1\pi}{a}\right)^2 + \left(\frac{m_2\pi}{b}\right)^2.$$

$$\phi\Big|_{z=0} = V_1\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \Rightarrow B_{m_1m_2} = V_1\delta_{m_1 1}\delta_{m_2 2}$$

$$\phi_{z=c} = V_2\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b} \Rightarrow \boxed{A_{22} = \frac{V_2}{\sinh K_{22}c}}$$

and $V_1\cosh K_{11}c + A_{11}\sinh K_{11}c = 0$

$$\boxed{A_{11} = -V_1\frac{\cosh K_{11}c}{\sinh K_{11}c}}$$

All other $A_{ij} = 0$

Aleiner Q5 solution

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a) The electric charge Q is uniformly distributed along the rod of the length L. Find the electric field at the distances from the rod (i) much larger than L; (ii) much smaller than L and near the middle of the rod; Solution: a) (i) At the distance $r >> L$ the field, $\vec{E}$ is not different from that of the point charge: The answer is then $\vec{E} = Q\vec{r}/r^3$; (ii) Let us chose the cylindrical coordinate system $(\vec{\rho}, z)$ such that the rod is characterized by $\vec{\rho} = 0; -L/2 < z < L/2$; The electric field near the middle of the rod is obviously directed along $\vec{\rho}$ One finds $\vec{E} = Q\vec{\rho} \int_{-L/2}^{L/2} \frac{dz}{L} \frac{1}{(\rho^2+z^2)^{3/2}}$; The integral is mainly contributed by $z \simeq \rho$, so that the limits of the integration can be put to infinity. We find $\vec{E} = 2Q\vec{\rho}(L\rho^2)$;

b) The same rod is now bent into the semicircle shape. Find the electric field in the center of the semicircle.

Chose the cylindrical coordinate system(z, ρ, ϕ) such that the rod is characterized by the equation $z = 0$; $\rho = \frac{L}{\pi}$; $\pi/2 < \phi < 3pi/2$ Electric field in the center is clearly along $\phi = 0$ direction and its absolute value is given by

$E = Q/L^2 \int_{\pi/2}^{3\pi/2} d\phi |\cos\phi|/pi = 2Q/(\pi L^2)$