Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 10, 2011
1:00PM to 3:00PM
Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

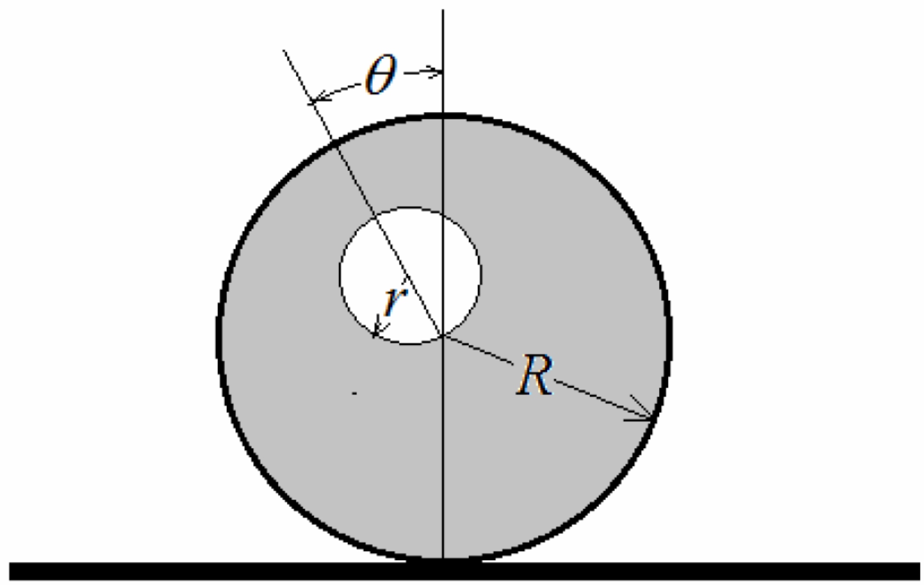
You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

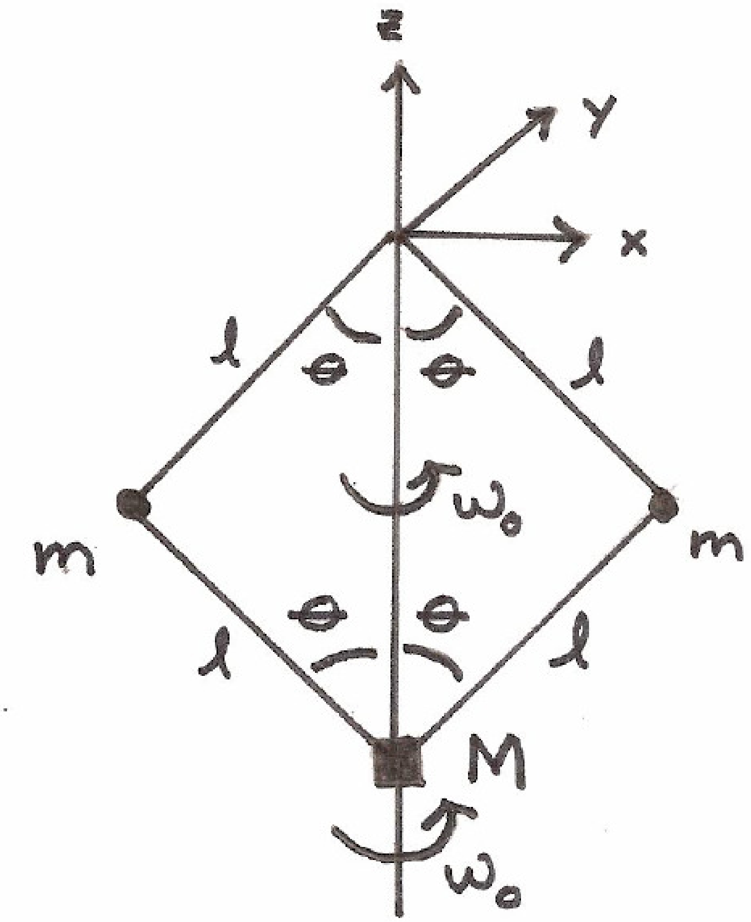
1. Begin with a solid cylinder of mass M and radius R. A hole is bored through this cylinder with radius $r < R/2$, parallel to the axis of the cylinder and with the surface of the hole touching the cylinder's axis. This modified cylinder then rolls on a horizontal surface under the influence of gravity. If the cylinder starts from rest with the orientation given by $\theta = \theta_0$, $|\theta_0| \ll 1$ find the subsequent motion.



2. A flyball governor consists of two masses m connected to arms of length l and a mass M as shown below. The assembly is constrained to rotate around a shaft on which the mass M can slide up and down without friction. Neglect the mass of the arms, air friction and assume that the diameter of mass M is small. Suppose that the shaft is constrained to rotate at angular velocity ω_0.

   (a) Calculate the equilibrium height of the mass M.

   (b) Calculate the frequency of small oscillations around this value.

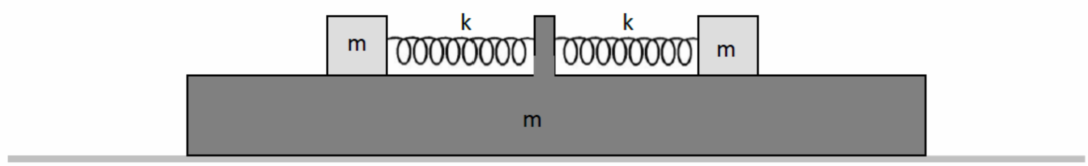


3. Consider a bucket of radius R that is hanging from a rope. The rope is twisted so that the bucket is spinning with a constant angular velocity ω about the rope, which defines a vertical axis through the center of the bucket. Write an equation which describes the shape of the surface of the water in the bucket as a function of r, the distance from the center of the bucket.

4. A platform of mass, m, is sitting on a frictionless surface and is free to move in the x-direction. Two identical blocks, also of mass, m, are connected to a post fixed to the platform by two identical springs of spring constant, k. The blocks are free to move on the platform in the x-direction with no friction.

   (a) Find the normal mode frequencies for the system in terms of k and m.

   (b) Describe the motion of each of the normal modes.



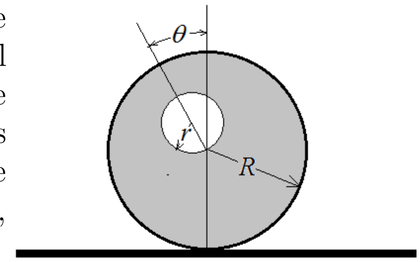
5. A pendulum consisting of a massless rod of length L with a mass M at its end hangs from a fixed point. A second pendulum of the same construction hangs from the end of the first pendulum. The pendula are constrained to move in the same plane.

   Assuming that neither pendulum moves far from the vertical, find the frequencies of small oscillation and clearly describe the corresponding motions.

N. Christ November 21, 2010

Quals Problem

1. Begin with a solid cylinder of mass M and radius R. A hole is bored through this cylinder with radius $r < R/2$, parallel to the axis of the cylinder and with the surface of the hole touching the cylinder's axis. This modified cylinder then rolls on a horizontal surface under the influence of gravity. If the cylinder starts from rest with the orientation given by $\theta = \theta_0$, $|\theta_0| \ll 1$ find the subsequent motion.



Suggested Solution (corrected)

1. (a) Use the coordinate θ to describe the cylinder's location.
   (b) The kinetic energy can be computed as $I\dot{\theta}^2/2$ where I is the moment of inertia about the point of contact P with the horizontal plane.
   (c) I can be obtained as the difference of the moment of inertia of the large cylinder about P minus that of the small cylinder. Using the parallel axis theorem:

$$I = (\frac{1}{2}+1)MR^2 - (\frac{1}{2}r^2 + (R+r)^2)(\frac{r}{R})^2 M \tag{1}$$

$$= \left(\frac{3}{2}R^2 - r^2 - \frac{\frac{3}{2}r^4 + 2r^3 R}{R^2}\right) M \tag{2}$$

   (d) To compute the increase in potential energy as a function of θ we can observe that the center of the hole falls by an amount $r(1-\cos(\theta))$ as θ increases from zero. This implies that the gravitational potential is $M(r/R)^2 gr(1-cos(\theta))$
   (e) Thus, in the small angle approximation the Lagrangian in terms of θ and $\dot{\theta}$ is

$$L = \frac{1}{2}I(\dot{\theta})^2 - \frac{1}{2}gM(r/R)^2 r\theta^2. \tag{3}$$

   (f) The system undergoes simple harmonic motion with frequency ω:

$$\omega = \sqrt{gM(r/R)^2 r/I} \tag{4}$$

   (g) and the motion is given by

$$\theta(t) = \theta_0 \cos(\omega t) \tag{5}$$

The x, y, z coordinates of masses m, m and M are $(-l\sin\theta, 0, -l\cos\theta)$, $(l\sin\theta, \theta, -l\cos\theta)$ and $(0, 0, -2l\cos\theta)$.

The velocity is given by $\dot{\vec{r}}' = \dot{\vec{r}} + \vec{\omega}_0 \times \vec{r}$

Note that $\vec{\omega}_0 = (0, 0, \omega_0)$.

The corresponding velocities are

$(-l\dot{\theta}\cos\theta, l\omega_0\sin\theta, l\dot{\theta}\sin\theta)$,

$(l\dot{\theta}\cos\theta, -l\omega_0\sin\theta, l\dot{\theta}\sin\theta)$,

$(0, 0, -2l\dot{\theta}\sin\theta)$

Kinetic energy is

$$T = ml^2\omega_0^2\sin^2\theta + ml^2\dot{\theta}^2 + 2Ml^2\dot{\theta}^2\sin^2\theta$$

Potential energy is

$$V = -2mgl\cos\theta - 2Mgl\cos\theta$$

$$L = T - V = ml^2\omega_0^2\sin^2\theta + ml^2\dot{\theta}^2 + 2Ml^2\dot{\theta}^2\sin^2\theta + 2(m+M)gl\cos\theta$$

Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$2(m + 2M\sin^2\theta)\lambda\ddot{\theta} - 2M\lambda\dot{\theta}^2\sin 2\theta$$

$$- m\lambda\omega_0^2\sin 2\theta + 2(m+M)g\sin\theta = 0$$

At equilibrium, $\ddot{\theta} = \dot{\theta} = 0$

and $\theta_0 = \theta$.

$$m\lambda\omega_0^2\sin 2\theta_0 = 2(m+M)g\sin\theta_0$$

Two solutions:

(i) $\theta_0 = 0$

(ii) $\cos\theta_0 = \dfrac{(m+M)g}{m\lambda\omega_0^2}$

The distance of mass M from the top are

(i) $\theta_0 = 0 \Rightarrow 2l$

(ii) $2\lambda\cos\theta_0 = \dfrac{2(m+M)g}{m\omega_0^2}$

(b) For small oscillations,

$\theta' = \theta - \theta_0$ and $\dot{\theta}' = \dot{\theta}$ and $\theta' \ll \theta_0$

$\sin\theta \approx \sin\theta_0 + \theta'\cos\theta_0$

$\sin 2\theta \approx \sin 2\theta_0 + 2\theta'\cos 2\theta_0$

Keeping only first order terms, the equation of motion becomes

$$2(m+2M\sin^2\theta_0)\,l\,\ddot{\theta}' - m l \omega_0^2 \sin 2\theta_0 - 2 m l \omega_0^2 \theta' \cos 2\theta_0$$
$$+ 2(m+M) g \sin\theta_0 + 2(m+M) g\, \theta' \cos\theta_0 = 0$$

2nd and 4th term cancel due to equilibrium condition.

$$(m+2M\sin^2\theta_0)\,l\,\ddot{\theta} + [(m+M)g\cos\theta_0 - m l \omega_0^2 \cos 2\theta_0]\,\theta = 0$$

Oscillation frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{(m+M)g\cos\theta_0 - m l \omega_0^2 \cos 2\theta_0}{(m+2M\sin^2\theta_0)\,l}}$$

Answer:

Analyze the problem in a non-inertial reference frame which is attached to the rotating bucket, with its y-axis directed vertically along the rope.

Consider forces on element of water of mass m on surface of water. In this frame, it is in static equilibrium, so forces much balance.

Vertical: (1) F cos(phi) – mg = 0, where F is normal force from other water, phi is angle with vertical (y) axis (and therefore also the angle between the surface of the water and the horizontal).

Radial: (2) F1 – F sin (phi) = 0, where F1 is the (fictitious) centrifugal force, given by F1 = mw^2r.

Solve (1), (2) to get phi, the angle of the surface with respect to the vertical:

tan (phi) = w^2r/g, which is the slope of the surface at position r (ie. it is dy/dr).

Integrate to get y(r) = integral (dy) = integral (w^2r/g) dr = ½ $w^2/g\ r^2$

So, the surface is quadratic in shape.

Solution:

With respect to a coordinate system fixed to the surface, let x = position of the platform, $x_1 =$ position of left block, $x_2 =$ position of right block.

Further, let the positive x-direction be to the right.

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}k\left(x - x_1\right)^2 - \frac{1}{2}k\left(x - x_2\right)^2$$

Lagrange's equations are:

$$m\ddot{x} + k\left(x - x_1\right) + k\left(x - x_2\right) = 0$$

$m\ddot{x}_1 - k\left(x - x_1\right) = 0$ and $m\ddot{x}_2 - k\left(x - x_2\right) = 0$

Assuming small oscillations with $x_i = A_i \cos\omega t$ gives

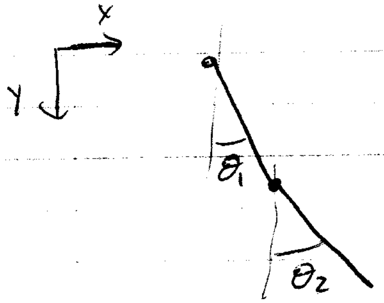
$$\begin{pmatrix} 2k - m\omega^2 & -k & -k \\ -k & k - m\omega^2 & 0 \\ -k & 0 & k - m\omega^2 \end{pmatrix}\begin{pmatrix} x \\ x_1 \\ x_2 \end{pmatrix} = 0$$

Setting the determinant equal to zero gives

$$-3k^2m\omega^2 + 4km^2\omega^4 - m^3\omega^6 = \omega^2\left(k - m\omega^2\right)\left(3k - m\omega^2\right) = 0$$

and yields normal mode freqencies of $0, \sqrt{k/m}$, and $\sqrt{3k/m}$

- For the 0 frequency mode, all masses move the same. $\left(x = x_1 = x_2\right)$
- For the $\sqrt{k/m}$ frequency mode, the platform is stationary and the blocks move equally in opposite directions. $\left(x = 0,\ x_1 = -x_2\right)$
- For the $\sqrt{3k/m}$ frequency mode, the two blocks move equally same direction and the platform moves in the opposite direction with twice the excursion. $\left(x = -2x_1 = -2x_2\right)$



$$x_1 = L\sin\theta_1$$
$$y_1 = L\cos\theta_1$$

$$x_2 = L(\sin\theta_1 + \sin\theta_2)$$
$$y_2 = L(\cos\theta_1 + \cos\theta_2)$$

$$KE = \frac{m}{2}\left\{L^2\dot\theta_1^2 + L^2\left[\dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2\cos(\theta_1 - \theta_2)\right]\right\}$$

$$PE = -mgL(2\cos\theta_1 + \cos\theta_2)$$

Assuming small angles,

$$Lag = \frac{mL^2}{2}\left[2\dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2\right]$$
$$+ mgL\left(2 - \frac{2}{2}\theta_1^2 - \frac{1}{2}\theta_2^2\right)$$

$$\Rightarrow \quad mL^2(2\ddot\theta_1 + \ddot\theta_2) = -2mgL\theta_1$$
$$mL^2(\ddot\theta_2 + \ddot\theta_1) = -mgL\theta_2$$

$$2\ddot\theta_1 + \ddot\theta_2 = -2\frac{g}{L}\theta_1$$
$$\ddot\theta_1 + \ddot\theta_2 = -\frac{g}{L}\theta_2$$

sol'ns: $\theta_1 = a_1\cos\omega t$
$\theta_2 = a_2\cos\omega t$

with

$$Ma = Ka$$

$$M = \begin{pmatrix} 2\omega^2 & \omega^2 \\ \omega^2 & \omega^2 \end{pmatrix}, \quad K = \begin{pmatrix} +2g/L & 0 \\ 0 & +g/L \end{pmatrix}$$

$$\Rightarrow \det\begin{pmatrix} 2\omega^2 - 2g/L & \omega^2 \\ \omega^2 & \omega^2 - g/L \end{pmatrix} = 0$$

$$\Rightarrow \omega^4 - 4\omega^2 \tfrac{g}{l} + 2\left(\tfrac{g}{l}\right)^2 = 0$$

$$\omega^2 = \left(2 \pm \sqrt{2}\right) \tfrac{g}{l}$$

$$\theta_1 \text{ eq} \Rightarrow 2\omega^2 a_1 + \omega^2 a_2 = 2 \tfrac{g}{l} a_1$$

$$\Rightarrow a_2/a_1 = 2\,\frac{\left(\tfrac{g}{l} - \omega^2\right)}{\omega^2}$$

$$= 2\,\frac{\left[1 - (2 \pm \sqrt{2})\right]}{(2 \pm \sqrt{2})}$$

$$= 2\,\frac{\left[1 - (2 \pm \sqrt{2})\right](2 \mp \sqrt{2})}{(4-2)}$$

$$= (2 \mp \sqrt{2}) - 2 = \mp\sqrt{2}$$

$$\omega^2 = (2 + \sqrt{2}) \tfrac{g}{l}$$

$$\theta_2 = -\sqrt{2}\,\theta_1$$

$$\omega^2 = (2 - \sqrt{2}) \tfrac{g}{l}$$

$$\theta_2 = \sqrt{2}\,\theta_1$$