Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 13, 2010
3:10 PM - 5:10 PM

Applied QM and Special Relativity
Section 4.

Two hours are permitted for the completion of this section of the examination. Choose **4 problems** out of the 5 included in this section. Remember to hand in **only** the 4 problems of your choice (if by mistake you hand in 5 problems, the highest scoring problem grade will be dropped). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Applied QM and Special Relativity), Question 2; Section 4 (Applied QM and Special Relativity), Question 6; etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 $1/2 \times 11$" paper (double-sided) you have prepared on Applied QM and Special Relativity. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are premitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. The hole spectrum of $GaAs$ at $k = 0$ is four-fold degenerate at $k = 0$ (Γ point of the Brillouin zone). In the vicinity of this point the spectrum is described by the Luttinger Hamiltonian

$$\hat{H} = Ak^2\hat{\mathbf{I}} + B(\vec{\mathbf{k}} \cdot \hat{\vec{\mathbf{J}}}\,)^{\mathbf{2}}$$

where $\hat{\mathbf{J}}_{x,y,z}$ are matrices of angular momentum $J = 3/2$ and $\hat{\mathbf{I}}$ is the unit matrix.

(a) Find the eigenvalues $\epsilon(k)$ of the Luttinger Hamiltonian.

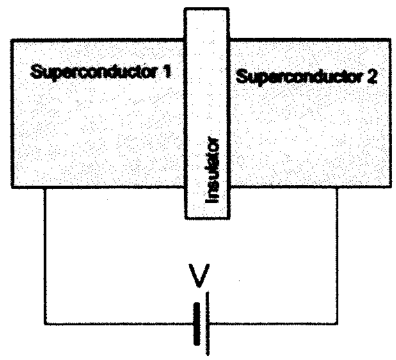
(b) The Luttinger Hamiltonian is spherically symmetric but the crystal has a cubic symmetry. Generalize the Luttinger Hamiltonian so it would have a cubic symmetry.

(c) If the crystal is deformed, the degeneracy at the Γ point can be partially lifted. What is the minimal possible degeneracy of the spectrum at the Γ point?

2. Two identical superconductors are separated by a thin insulator and connected to a battery whose DC voltage is given by V, as shown in the figure below. Let ψ_1 be the wave function of the condensed superconducting electron pairs on one side of the superconductor and ψ_2 be the wave function on the other side. The two wave functions are related to each other by the time dependent Schrödinger equation in the following way:

$$i\hbar\frac{\partial}{\partial t}\psi_1 = eV\psi_1 + K\psi_2$$

$$i\hbar\frac{\partial}{\partial t}\psi_2 = -eV\psi_2 + K\psi_1$$



Here, the constant K is a characteristic of junctions, related to the tunneling process of the electron pairs across the insulator, and V is the voltage applied by the battery.

In this problem we express each wave function in terms of its corresponding condensation density and the phase of the wave function: $\psi_1 = \sqrt{n_1}e^{i\theta_1}$ and $\psi_2 = \sqrt{n_2}e^{i\theta_2}$, where n_1 and n_2 are the densities, and θ_1 and θ_2 are the phases of the condensate wave functions of superconductor 1 and 2, respectively.

(a) Assuming n_1 and n_2 are real, show that the current density of this junction is given by

$$J = \frac{\partial n_1}{\partial t} = -\frac{\partial n_2}{\partial t} = J_0 \sin\delta$$

where $\delta = \theta_2 - \theta_1$. Find the expression for J_0 in terms of K, n_1, and n_2.

(b) Assume that initially the condensation densities are equal and large, and that the tunneling probability is small so that $n_1(t) \approx n_2(t)$. Show that the current density J derived in part (a) oscillates periodically over time. Find the frequency of the oscillation in terms of the applied DC voltage V.

3. A spinless particle of charge $-e$ and mass m is constrained to move in the x-y plane. There is a constant magnetic field $\vec{\mathbf{B}}$ along the direction normal to the plane. Assume that the field derives from a vector potential that has a single component along the x-direction given by $\mathrm{A}_x = -\mathrm{B}y$.

   (a) Write the expression for the Hamiltonian of one particle.

   (b) To find the solutions of the Schrödinger equation for the stationary states, consider wavefunctions
   $$\psi(x, y) = f(x)\phi(y)$$
   where
   $$f(x) = \exp\left[(i/\hbar)p_x x\right]$$
   and p_x is the x-component of momentum.

   Write the Schrödinger equation for $\phi(y)$ and obtain the expression for the spectrum of energy levels E_n (Landau levels) in the field $\vec{\mathbf{B}}$. What are the quantum numbers that correspond to a Landau level?

   (c) Assume that the area of the plane is given by the product of two lengths $L_x L_y$, that are along the x- and y-directions. Also assume that the function $f(x)$ satisfies the 'obvious' boundary condition
   $$f(x = 0) = f(x = L_x).$$

   Find the degeneracy of a Landau level as a function of the magnetic field for $L_x = L_y = L$.

4. A perpendicularly incident monochromatic plane wave is reflected from a mirror moving with a constant velocity along the line of propagation of the wave. Using Maxwell's equations, determine the angular frequency of the reflected wave as seen by a stationary observer.

5. In colliding beam detectors, K^0_{short} mesons can be detected through their decay to two charged pions

$$K^0_{\text{short}} \rightarrow \pi^+\pi^-$$

Cylindrical gas trackers composed of many wires in an argon gas volume located inside a solenoidal magnet can detect the ionization trail left by the pions and measure their vector momenta.

The lifetime of the K^0_{short} is 0.89×10^{-10} s and the mass is 498 MeV. (The mass of the charged pion is 140 MeV.)

For the following questions, assume that the energy of the K^0_{short} in the laboratory frame of the detector is 60 GeV.

(a) What is the minimum opening angle in the lab frame of the two pions from the K^0_{short} decay?

(b) How far, on average, does the K^0_{short} go before decaying into two pions?

(c) How far, on average, would the K^0_{short} go before interacting with an argon atom in the gas if the cross section for K+p or K+n interactions is about 20 millibarns (1barn = 10^{-28} m^2)? (The density of argon gas is 1.8×10^{-3} g/cm^3.)

(d) The K^0_{long} has a lifetime of 5.17×10^{-8} s and a substantial fraction (38.7%) decay as

$$K^0_{\text{long}} \rightarrow \pi^{\pm}e^{\pm}\nu_e$$

From this information, what branching fraction would you predict for the decay

$$K^0_{\text{short}} \rightarrow \pi^{\pm}e^{\pm}\nu_e$$

AQM Aleiner
Sec. 4
Rel + Applied QM
1

The hole spectrum of $GaAs$ at $k = 0$ is four-fold degenerate at $k = 0$ (Γ point of the Brillouin zone). In the vicinity of this point the spectrum is described by the Luttinger Hamiltonian

$$\hat{H} = Ak^2\hat{I} + B(\vec{k} \cdot \hat{\vec{J}})^2$$

where $\hat{J}_{x,y,z}$ are the matrices of angular momentum $J = 3/2$ and $\hat{I}$ is the unit matrix.

1. Find the eigenvalues $\epsilon(k)$ of the Luttinger Hamiltonian.
2. The Luttinger Hamiltonian is spherically symmetric and the crystal has a cubic symmetry. Generalize the Luttinger Hamiltonian so it would have a cubic symmetry as well.
3. If the crystal is deformed, the degeneracy at Γ point can be partially lifted. What is the minimal possible degeneracy of the spectrum in Γ point?

Solution:

1. Choose direction of k as z-axis. Then

$$\epsilon(k; J_z = \pm 1/2) = k^2(A + B/4); \quad \text{light holes}$$

and

$$\epsilon(k; J_z = \pm 3/2) = k^2(A + 9B/4); \quad \text{heavy holes.}$$

2.

$$\hat{H} = Ak^2\hat{I} + B(\vec{k} \cdot \hat{\vec{J}})^2 + C(k_x^2\hat{J}_x^2 + k_y^2\hat{J}_y^2 + k_z^2\hat{J}_z^2)$$

3. As the electron has spin 1/2 and the time reversal symmetry is not broken the minimal degeneracy in Γ-point is two because of the Kramers theorem.

AQM Kim

Sec. 4
Relativity + Applied QM
2

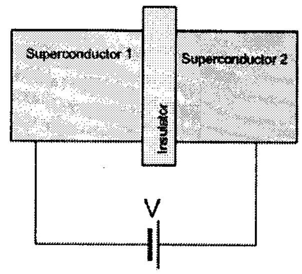
Applied QM

DC Josephson superconductor tunneling

Two identical supercondutors are separated by a thin insulator and connected to a battery whose DC voltage is given by V as shown in the figure below. Let ψ_1 be the wave function of the condensed superconducting electron pairs in one side of super conductor and ψ_2 be the wave function of the other side. The two wave functions are related to each other by the time-dependent Schrödinger equation in the following way:

$$i\hbar\frac{\partial}{\partial t}\psi_1 = eV\psi_1 + K\psi_2$$

$$i\hbar\frac{\partial}{\partial t}\psi_2 = -eV\psi_2 + K\psi_1$$



Here, the constant K is a characteristic of junctions related to the tunneling process of the electron pairs across the insulator and V is voltage applied by the battery outside.

In this problem we express each wav function in its corresponding condensation density and the phase of wave function: $\psi_1 = \sqrt{n_1}e^{i\theta_1}$ and $\psi_2 = \sqrt{n_2}e^{i\theta_2}$ where n_1 and n_2 are the density of condensate and θ_1 and θ_2 are the phase of the condensate wave functions of superconductor 1 and 2, respectively.

(a) Considering n_1 and n_2 are real, show that the current density of this junction defined is given by

$$J = \frac{\partial n_1}{\partial t} = -\frac{\partial n_2}{\partial t} = J_0 \sin\delta$$

where $\delta = \theta_2 - \theta_1$. Find the expression of J_0 in terms of K and n_1 and n_2.

(b) We assume that initially the condensation densities are equal and large, and further assume that the tunneling probability is small so that$_2 n_1 \approx n$ for all time.S how that the current density J derived above oscillates periodically over time. Find the frequency of the oscillation in terms of the applied DC voltage V.

Applied QM

DC Josephson tunneling

(a) $$\psi_1 = \sqrt{n_1}\, e^{i\theta_1} \Rightarrow \frac{\partial}{\partial t}\psi_1 = \frac{1}{2}\frac{\dot{n}_1}{\sqrt{n_1}} e^{i\theta_1} + i\dot{\theta}_1 \sqrt{n_1}\, e^{i\theta_1}$$

$$\psi_2 = \sqrt{n_2}\, e^{i\theta_2} \Rightarrow \frac{\partial}{\partial t}\psi_2 = \frac{1}{2}\frac{\dot{n}_2}{\sqrt{n_2}} e^{i\theta_2} + i\dot{\theta}_2 \sqrt{n_2}\, e^{i\theta_2}$$

From the coupled eqn.

$$\frac{i\hbar}{2}\frac{\dot{n}_1}{\sqrt{n_1}} e^{i\theta_1} - \hbar\dot{\theta}_1 \sqrt{n_1} e^{i\theta_1} = \frac{2eV}{2}\sqrt{n_1} e^{i\theta_1} + K\sqrt{n_2} e^{i\theta_2}$$

or $$\frac{i\hbar}{2}\frac{\dot{n}_1}{n_1} - \hbar\dot{\theta}_1 = eV + K\sqrt{\frac{n_2}{n_1}}\, e^{i\delta} \quad —①$$

Like wise we have

$$i\hbar\frac{\dot{n}_2}{n_1} - \hbar\dot{\theta}_2 = -eV + K\sqrt{\frac{n_1}{n_2}}\, e^{-i\delta} \quad —②$$

Considering n_1, n_2, θ_1 & θ_2 are all real function, from the imaginary part of ① & ② we have

$$\frac{\hbar}{2}\frac{\dot{n}_1}{n_1} = K\sqrt{\frac{n_2}{n_1}}\sin\delta, \qquad \frac{\hbar}{2}\frac{\dot{n}_2}{n_2} = -K\sqrt{\frac{n_1}{n_2}}\sin\delta$$

$$\Rightarrow \boxed{J = \dot{n}_1 = -\dot{n}_2 = \underbrace{\frac{2}{\hbar}K\sqrt{n_1 n_2}}_{= J_0}\sin\delta}$$

(b) From the real part of ① & ②, & $n_1 \simeq n_2$

$$-\hbar\dot{\theta}_1 = eV + K\sqrt{\frac{n_2}{n_1}}\cos\delta \simeq eV + K\cos\delta$$

$$\Rightarrow \frac{d\delta}{dt} = \dot{\theta}_2 - \dot{\theta}_1 = \frac{2e}{\hbar} V$$

or $$\boxed{\delta(t) = \delta_0 + \frac{2eV}{\hbar} t}$$

From (a)

$$J(t) = \frac{2}{\hbar} K \sqrt{n_1 n_2} \sin\Big[\delta_0 + \underbrace{\frac{2eV}{\hbar}}_{=\omega} t\Big]$$

$$\Rightarrow \boxed{\hbar\omega = 2eV}$$

//

Pinczuk
Sec. 4 Pet r
Appl. QM
#3

General-Section 4: applied quantum mechanics

A spin-less particle of charge –e and mass m is constrained to move in the x-y plane. There is a constant magnetic field B along the direction normal to the plane. Assume that the field derives from a vector potential that has a single component along the x-direction given by $A_x = -By$.

(a) write the expression for the Hamiltonian of one particle.

(b) to find the solutions of the Schroedinger equation for the stationary states consider wavefunctions

$\psi(x,y)= f(x)\phi(y)$

where

$f(x)= \exp[(i/\hbar)p_x x]$

and p_x is the x-component of momentum.

Write the Schroedinger equation for $\phi(y)$ and obtain the expression for the spectrum of energy levels E_n (Landau levels) in the field B.
What are the quantum numbers that correspond to a Landau level?

(c) Assume that the area of the plane is given by the product of two lengths L_xL_y, that are along the x- and y-directions.
Also assume that the function f(x) satisfies the 'obvious' boundary condition

$f(x=0)=f(x=L_x)$

Find the degeneracy of a Landau level as function of magnetic field for $L_x=L_y=L$.

Solution

(a) $$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2$$

$\vec{A} = (A_x, 0, 0) \qquad A_x = -By$

$$H = \frac{1}{2m}\left(p_x + \frac{e}{c}By\right)^2 + \frac{1}{2m}p_y^2$$

(b) $$\frac{d^2\phi(y)}{dy^2} + \frac{2m}{\hbar^2}\left[E_n - \frac{1}{2}m\omega_c^2(y-y_0)^2\right]\phi(y) = 0$$

$$\omega_c = \frac{eB}{mc} \qquad |y_0| = \frac{cp_x}{eB}$$

$$E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega_c$$

(c) From the boundary condition

$e^{(i/\hbar)p_x L} = 1$; which implies

$$p_x = m\,2\pi\frac{\hbar}{L}, \quad m = \text{integer}$$

$$y_0 = 2\pi\frac{c\hbar}{eBL}\,m \le L$$

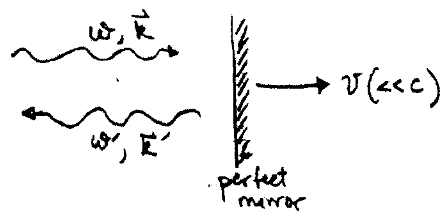
The number of states is equal to m for the maximum value $y_0 = L$

$$N = (hc/eB)A \;;\; A = L^2$$

Marka
Sec. 4 Rel + App QM
4

#5: A perpendicularly incident monochromatic plane wave is reflected from a mirror moving with a constant velocity along the line of the propagation of the wave. Using Maxwell's equations, determine the angular frequency of the reflected wave as seen by a stationary observer.

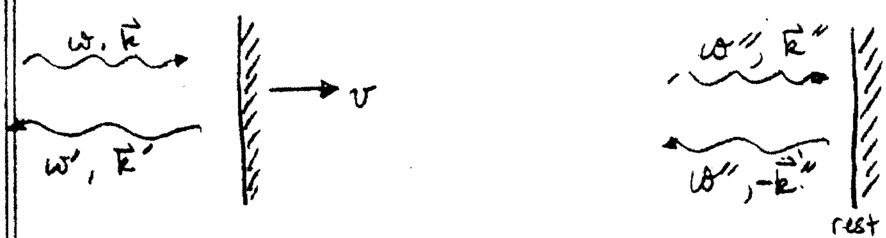
Solution from Gabe-Perez Giz:



This problem has a relativistic solution and can be done with Lorentz transformations. The hint, however, suggests a solution based on the principles of applying BCs at the interface b/w media. First, I'll do the relativistic solution with 4-vectors (it was not expected that you would do the problem this way), and then I'll redo the problem by applying boundary conditions to the waves. Remarkably (or perhaps not remarkably), we will get the same exact answer, and thus the same 1st order approximation (or n^{th} order approximation, for that matter), as well.

SOLUTION W/ LORENTZ TRANSFORMATIONS

LAB FRAME S   MIRROR FRAME S′



In the rest frame of the mirror, we know what happens b/c it is a standard reflection problem of the sort we've seen many times: the incident and reflected waves have the same frequency (which I've called ω'') and their wavevectors differ only in direction (opposite directions, but $|\vec{k}''| = |-\vec{k}''| = \omega''/c$).

To translate this into the frequencies in the lab frame, it's useful to know that ω and $\vec{k}$ form a 4-vector:

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} \quad \ldots \text{ so the 4-vector can be written compactly as } (\omega/c, \vec{k}).$$

NOTE: You can somewhat understand WHY ω/c & $\vec{k}$ make a 4-vector by thinking of individual photons, for which $\begin{pmatrix} \hbar\omega/c \\ \hbar\vec{k} \end{pmatrix} = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} =$ 4-momentum of the photon, which is a 4-vector. So $\begin{pmatrix} \omega/c \\ \vec{k} \end{pmatrix} \equiv k^{\mu} = \frac{1}{\hbar} p^{\mu}$ and is itself a 4-vector.

Strategy: write down $k^{\mu}_{incident}$ in the lab frame. Get $k^{\mu'}_{incident}$ in S' via Lorentz transformation; get $k^{\mu'}_{ref}$ in S' by inspection; get

k^μ_{ref} in LAB frame by inverse Lorentz transformation. Then read off the frequency component of k^μ_{ref}.

$$k^\mu_{inc} = \begin{pmatrix} \omega/c \\ 0 \\ 0 \\ |\vec{k}| \end{pmatrix}$$ if the waves and mirror all move in the $\hat{z}$ direction.

By the way, $|\vec{k}| = \omega/c$ for electromagnetic waves propagating in vacuum, so we write k^μ_{inc} more compactly as

$$k^\mu_{inc} = \omega/c \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Now, do a Lorentz transformation to the components of this 4 vector in the mirror frame:

$$\underbrace{k^{\mu'}_{inc}}_{\text{mirror frame}} = \omega/c \begin{pmatrix} \gamma[1-\frac{v}{c}(1)] \\ 0 \\ 0 \\ \gamma[1-\frac{v}{c}(1)] \end{pmatrix} = \omega/c\,\gamma \begin{pmatrix} 1-v/c \\ 0 \\ 0 \\ 1-v/c \end{pmatrix}$$

By arguments made on the prior page, we now know $k^{\mu'}_{ref}$ by inspection: same ω, opposite $\vec{k}$ ...

$$k^{\mu'}_{ret} = \frac{\omega}{c}\gamma \begin{pmatrix} 1-\frac{v}{c} \\ 0 \\ 0 \\ \frac{v}{c}-1 \end{pmatrix}$$

Now, do an <u>inverse</u> Lorentz transformation to get the components of k^μ_{ref} in the LAB frame:

$$k^\mu_{ret} = \frac{\omega}{c}\gamma \begin{pmatrix} \gamma[(1-\frac{v}{c}) + \frac{v}{c}(\frac{v}{c}-1)] \\ 0 \\ 0 \\ \gamma[(\frac{v}{c}-1) + \frac{v}{c}(1-\frac{v}{c})] \end{pmatrix} = \frac{\omega}{c}\gamma^2 \begin{pmatrix} (1-\frac{v}{c})^2 \\ 0 \\ 0 \\ -(1-\frac{v}{c})^2 \end{pmatrix}$$

The timelike component of k^{μ}_{ref} IS the frequency of the reflected wave in the lab frame, so we get

$$\frac{\omega'}{c} = \frac{\omega}{c}\underbrace{\gamma^2}_{\frac{1}{1-(v/c)^2}}\left(1-\frac{v}{c}\right)^2 = \frac{\omega}{c}\,\frac{\left(1-\frac{v}{c}\right)^2}{1-\left(\frac{v}{c}\right)^2} = \frac{\omega}{c}\,\frac{(1-v/c)(1-v/c)}{(1-v/c)(1+v/c)}$$

$$= \frac{\omega}{c}\,\frac{1-v/c}{1+v/c} \Rightarrow \boxed{\omega' = \omega\,\frac{1-v/c}{1+v/c}}$$

We're told $v \ll c$, so we want to expand to lowest order in v/c. Use a binomial expansion: $(1+x)^n \simeq 1 + nx$ to 1st order. Thus,

$$\left(1+\frac{v}{c}\right)^{-1} \simeq 1-\frac{v}{c} \quad \text{to 1st order}, \quad \text{so}$$

$$\omega' \simeq \omega\left(1-\frac{v}{c}\right)\left(1-\frac{v}{c}\right) = \omega\Big(1-2\frac{v}{c}+\underbrace{\frac{v^2}{c^2}}_{\substack{\text{neglect b/c} \\ \text{we are only} \\ \text{keeping lowest order in } v/c}}\Big) = \boxed{\omega\left(1-2\frac{v}{c}\right)}$$

<u>SOLUTION W/ PLANE WAVES and BC's</u>

Forget relativity... let's just say you have an incident and a reflected wave (but no transmitted wave, since the mirror is, say, a perfect conductor) of <u>different frequencies</u> at the mirror surface. Since the waves are traveling in the z-direction and normally incident on the mirror, then the waves are polarized parallel to the surface of the mirror. For simplicity, let's say the incident wave has a single polarization (and ergo, from HW3, the reflected wave has the same polarization), and called the direction of polarization the $\hat{x}$ direction for ease.

Let's match BC's at the surface of the mirror:

$$\tilde{\vec{E}}_I = \tilde{E}_{0_I}\, e^{i(kz-\omega t)}\,\hat{x} \qquad \tilde{\vec{E}}_T = 0$$
$$\tilde{\vec{E}}_R = \tilde{E}_{0_R}\, e^{i(-k'z-\omega' t)}\,\hat{x}$$

<u>NOTE</u> that the waves have different ω's and thus must have different k's (since $\frac{\omega}{k} = \frac{\omega'}{k'} = c$). Also, <u>NOTE</u> that I've put a $-k'$ in the reflected wave to capture the fact that it moves in the negative z-direction.

There is no component of $\tilde{\vec{E}}$ normal to the surface of the boundary b/c $\tilde{\vec{E}}$ is polarized $\parallel$ boundary. Also, the BC's on $\tilde{\vec{B}}$ will give no new information beyond what the $\tilde{\vec{E}}$ BC's give since all the nonzero waves are in the same medium (vacuum)

$$\Rightarrow \text{BC}: \quad \tilde{E}_{0_I}\, e^{i(kz-\omega t)} + \tilde{E}_{0_R}\, e^{i(-k'z-\omega' t)} = 0$$

At the boundary and for all time.

Say the mirror passes the $z=0$ plane at $t=0$. Then the boundary is at $\underline{z = vt}$. Insert that above and also write $k = \omega/c$ and $k' = \omega'/c$:

$$\Rightarrow \tilde{E}_{0_I}\, e^{i(\omega \frac{v}{c} t - \omega t)} + \tilde{E}_{0_R}\, e^{i(-\omega' \frac{v}{c} t - \omega' t)} = 0$$

Now, this needs to hold <u>FOR ALL</u> t. Following the logic we've used repeatedly in class and HW for applying BC's, we see that this can only happen (since $\tilde{E}_{0_I}$ and $\tilde{E}_{0_R}$ do not depend on t) if the arguments of the exponentials are <u>identical</u>:

$$\frac{\omega v}{c}\cancel{t} - \omega \cancel{t} = -\frac{\omega' v}{c}\cancel{t} - \omega' \cancel{t}$$

$$\omega\left(\frac{v}{c} - 1\right) = -\omega'\left(\frac{v}{c}+1\right)$$

$$\Rightarrow \omega' = \omega \frac{1 - v/c}{1 + v/c} \simeq \omega\left(1 - 2\frac{v}{c}\right),$$ exactly as before!

Pretty awesome way to do this problem, no?

Relativity - Shaevitz
Sec 4 Rel & Appl QM

Quals Problem 2 – Relativity

M. Shaevitz
Fall, 2009

5

In colliding beam detectors, K^0_{short} mesons can be detected through their decay to two charged pions

$$K^0_{short} \to \pi^+\pi^-$$

Cylindrical gas trackers composed of many wires in an argon gas volume located inside a solenoidal magnet can detect the ionization trail left by the pions and measure their vector momenta.

The lifetime of the K^0_{short} is 0.89×10^{-10} s and the mass is 498 MeV. (The mass of a charged pion is 140 MeV.)

For the following questions, assume that the energy of the K^0_{short} in the laboratory frame of the detector is 60 GeV.

a) What is the minimum opening angle in the lab frame of the two pions from the K^0_{short} decay?

b) How far on average does the K^0_{short} go before decaying into the pions?

c) How far on average would a K^0_{short} go before interacting with an argon atom in the gas if the cross section for K+p or K+n interactions is about 20 millibarns? (The density of argon gas is 1.8×10^{-3} g/cm^3.)

d) The K^0_{long} has a lifetime of 5.17×10^{-8} s and a substantial (38.7 %) decay fraction to

$$K^0_{long} \to \pi^\pm e^\mp \nu_e$$

From this information, what branching fraction would you predict for the

$$K^0_{short} \to \pi^\pm e^\mp \nu_e$$

Solution:

a) Minimum opening angle when $\theta_{cm} = 90°$

$$\gamma_{K^0} = \frac{E_{K^0}}{M_{K^0}} = \frac{60\,GeV}{0.498\,GeV} = 120.5 \qquad \beta_{K^0} \approx 1$$

In K^0 rest frame for $\theta_{cm} = 90°$ $\Rightarrow$ $E^{\pi} = \frac{m_{K^0}}{2}$ $\quad P_{\perp}^{\pi} = \left(\left(\frac{m_{K^0}}{2}\right)^2 - m_{\pi}^2\right)^{1/2}$

Boost to lab

$$P_{\perp}^{LAB} = P_{\perp}^{cm} = \left(\left(\frac{m_{K^0}}{2}\right)^2 - m_{\pi}^2\right)^{1/2} = 0.206\ GeV$$

$$P_{\parallel}^{LAB} = \gamma\left(E_{cm}^{\pi} + P_{\parallel cm}^{\pi}\right) = \frac{E_{K^0}}{m_{K^0}}\left(\frac{m_{K^0}}{2}\right) = \frac{E_{K^0}}{2}$$

$$= 30\ GeV$$

$$\tan\theta = \frac{0.206\,GeV}{30\,GeV} = 0.00687 \Rightarrow \theta = 0.00687 = 6.9\,mr$$

$$\theta_{opening} = 2\theta = 13.7\,mr$$

b) $d = \gamma c \tau = (120.5)(3\times10^{8}\,m/s)(0.89\times10^{-10}\,s)$

$= 3.21\,m$

c) $L_{int} = \frac{1}{\rho\,\sigma\,N_{avg}} = \frac{1}{(1.8\times10^{-3}\,g/cm^3)(2\times10^{-26}\,cm^2)(6.02\times10^{23}/g)}$

$= 461\,m$

d) $BR_{Ke3}^{Kshort} = \frac{\Gamma_{Ke3}}{\Gamma_{TOTAL}^{Kshort}}$ $\qquad BR_{Ke3}^{KLONG} = \frac{\Gamma_{Ke3}}{\Gamma_{TOTAL}^{KLONG}} \Rightarrow \Gamma_{Ke3} \propto \frac{BR_{Ke3}^{KLONG}}{\tau_{KLONG}}$

$$\Gamma_{TOTAL}^{Kshort} \propto \frac{1}{\tau_{Kshort}}$$

$$BR_{Ke3}^{Kshort} = \frac{BR_{Ke3}^{KLONG}\cdot\tau_{Kshort}}{\tau_{KLONG}} = \frac{(0.387)(0.89\times10^{-10}\,s)}{(5.17\times10^{-8}\,s)} = 6.7\times10^{-4}$$