Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 12, 2009
1:00 PM - 3:00 PM

Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 1; Section 1 (Classical Mechanics) Question 3, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

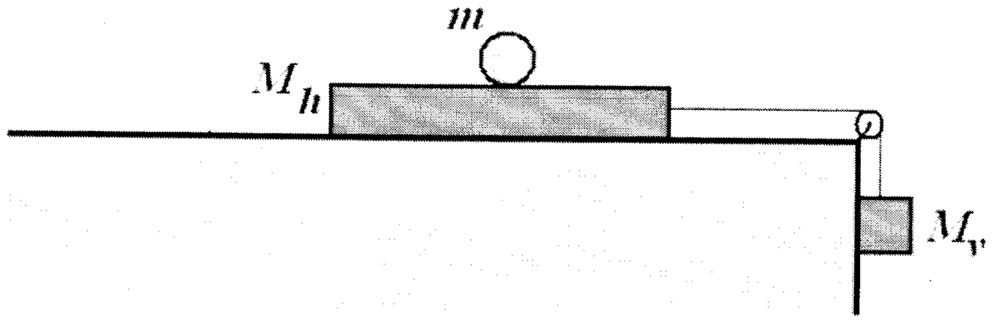
You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

1. A block of mass M_h slides without friction on a horizontal table. It is connected by a massless rope passing over a massless frictionless pulley to a second hanging mass M_v pulled downward by gravity. A sphere of mass m and radius R, initially at rest, rolls without sliding on the top surface of the first block. Find the resulting acceleration of the mass M_v and the center of mass of the sphere.

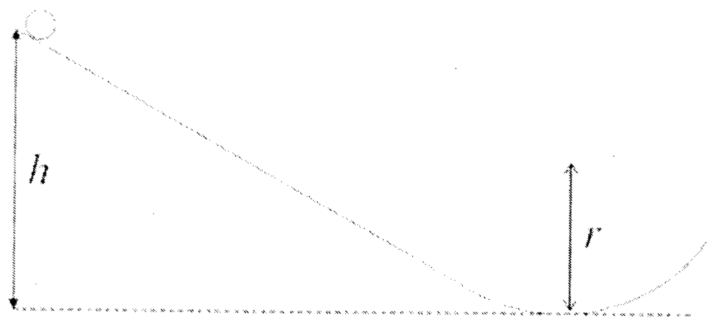


2. A typical disk drive has platters of mass $m = 20$ g, radius $R = 4$ cm, and thickness $t = 1$ mm rotating around a central shaft ("spindle"). New disk drives have platters rotating at frequencies ≥ 10 kHz and at such high frequencies, a small misalignment of the rotation axis and the shaft can cause serious mechanical problems. ***Estimate*** the torque around the center of mass of the platter required to keep a single platter rotating at a frequency $f = 10$ kHz when the shaft is tilted by an angle of 0.1 mrad (0.0001 rad) from the angular velocity vector.

Hint: you may use the fact that the moment of inertia of a disk about an axis in the plane of the disk through the center is one half (½) of the moment of inertia about an axis normal to the disk and through the center.

3. A non-uniform ball of mass M and radius R rolls smoothly from rest down a ramp and onto a circular loop of radius r. The ball is initially at a height h above the bottom of the loop. At the bottom of the loop, the normal force on the ball is twice its weight.

Expressing the rotational inertia of the non-uniform ball in the general form $I = \beta M R^2$, determine an expression for β in terms of h and r.



4. A particle of mass m moves in the 2-dimensional potential

$$V = \frac{1}{2} k \sin^2(\sqrt{x^2 + y^2 - xy})$$

(a) Write the Lagrangian for the system.

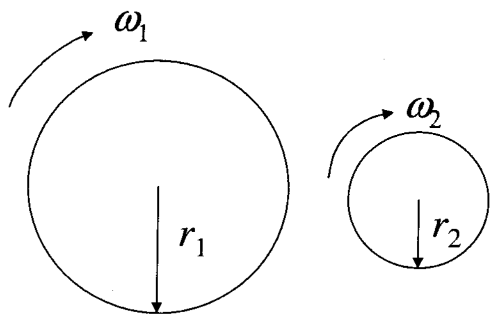
(b) Is the origin a stable equilibrium for the particle?

(c) Write the Lagrangian appropriate for small oscillations about $x = y = 0$.

(d) Calculate the normal frequencies.

(e) Write the general small oscillation solution and sketch the normal modes.

5. Two uniform cylinders spin independently about their axes (the axes are parallel to each other). The first has radius r_1 and mass m_1, the other has radius r_2 and mass m_2. Initially they rotate in the same sense of rotation with angular speeds ω_1 and ω_2 respectively. They are then brought together so that they touch. After the steady state is achieved, what is the final angular velocity of cylinder 1, ω_1'?



Christ sec 1 Mechanics #1

Suggested Solutions

1.
- Introduce the tension T in the rope and the force F which the mass M_h exerts to the right on the sphere. Use X, Y and x for the laboratory coordinates of M_h, M_v and m respectively.
- Write down equations for the acceleration of the cm of each mass:

$$\begin{aligned} M_h\ddot{X} &= T - F \\ m\ddot{a} &= F \\ M_v\ddot{Y} &= T - M_v g \end{aligned}$$

- Let θ represent the angular orientation of the sphere (increasing with clockwise motion) and write the equation for the angular acceleration of the sphere and the relation between $\ddot{\theta}$, $\ddot{a}$ and $\ddot{X}$:

$$\begin{aligned} \frac{2}{5}mR^2\ddot{\theta} &= -FR \\ \ddot{a} &= \ddot{X} + R\ddot{\theta} \end{aligned}$$

- These five equations can then be solved for T, F, $\ddot{\theta}$, $\ddot{X}$, $\ddot{Y}$,

2. [redacted]

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Cole Sec 1 MECH
Problem # 2

Qualifying exam solutions: Mechanics, Disk Drive Problem

In these solutions I will take the axis of the angular momentum vector as the inertial z axis. Then, the angular momentum should be evaluated in body-centered coordinates using principle axes defined as follows:

- z′ axis normal to disk through the center
- x′ axis in plane of the disk through the center and lying in the plane containing the z and z′ axes
- y′ normal to the others

Then, the total angular momentum in body centered coordinates is

$$\vec{L}_{bc} = I_1 \omega'_x \hat{x}' + I_2 \omega'_y \hat{y}' + I_3 \omega'_z \hat{z}'$$

where ω'_x, ω'_y, ω'_z are the components of the angular velocity vector projected onto the body-centered, principle axes. If we write the angle between the angular velocity vector and the normal to the disk as β, then

$$\omega'_z = \omega_z \cos\beta = \omega \cos\beta \approx \omega$$

and

$$\omega'_x = \omega_z \sin\beta = \omega \sin\beta \approx \omega\beta$$

while $\omega'_y = 0$. Now,

$$I_3 = \frac{1}{2} m R^2$$

and using the hint,

$$I_1 = \frac{1}{4} m R^2$$

So, the body-centered angular momentum is

$$\vec{L}_{bc} = \frac{1}{4} m R^2 \omega \left(\beta \hat{x}' + 2\hat{z}' \right)$$

Because the disk is rotating, the angular momentum vector will rotate. We can evaluate the time derivative of the angular momentum vector in inertial coordinates using

$$\dot{\vec{L}}_{in} = \vec{\omega} \times \vec{L}_{bc} + \dot{\vec{L}}_{bc}$$

for which the second term is zero in this problem. Expressing the angular velocity in body centered coordinates as above, we have

$$\dot{\vec{L}}_{in} = -\frac{1}{4} m R^2 \omega^2 \beta \hat{y}'$$

The torque is equal to the time derivative of the angular momentum, so

$$|\vec{\tau}| = \frac{1}{4} m R^2 \omega^2 \beta$$

But this expression needs to be expressed in terms of the rotational frequency, not the angular frequency, so

$$|\vec{\tau}| = m\beta(\pi R f)^2$$

Plugging in numbers, $\pi Rf \approx 1.25\text{x}10^5$ cm s^{-1} = $1.25\text{x}10^3$ m s^{-1}. Then

$|\tau| \approx 0.02$ kg x $1\text{x}10^{-4}$ x $1.6\text{x}10^6$ m^2s^{-2} = 3 n m.

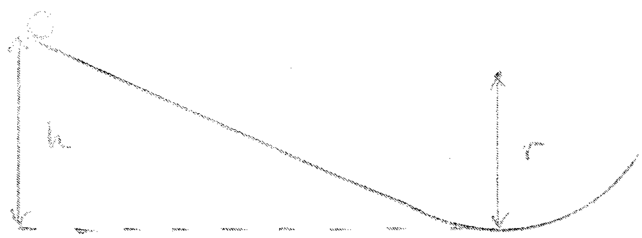
Sec 2
Dodd - Mechanics
Problem # 3

Quals Exam Questions (Dodd)

Mechanics Q

A non-uniform ball of mass M and radius R rolls smoothly from rest down a ramp and onto a circular loop of radius r. The ball is initially at a height h above the bottom of the loop. At the bottom of the loop, the normal force on the ball is twice its weight.

Expressing the rotational inertia of the non-uniform ball in the general form $I = \beta MR^2$, determine an expression for β in terms of h and r.



Solution:

As the ball rolls around the circular loop at the bottom of the track, it experiences centripetal acceleration (v^2/r) which we find from:

$$N - W = \frac{Mv^2}{r}$$

where N is the normal force and W is the weight. We are given than $N = 2Mg$, and so:

$$2Mg - Mg = \frac{Mv^2}{r}$$

i.e. $v^2 = gr$. Relating the angular and translational velocities by $v=r\omega$, we next use the expression for total kinetic energy of a rolling object (no slipping), viz:

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

and apply energy conservation for the ball between its initial position at rest and its position at the bottom of the loop:

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

Dodd

Sec 1
MECH
Problem #3
page 2

Substituting for v and for ω (from above), and also for $I = \beta M R^2$, we find after some simple manipulation:

$$h = \frac{1}{2}r + \frac{1}{2}\beta r$$

and re-arranging:

$$\beta = \frac{2h}{r} - 1$$

A. Mueller

Mechanics

A particle of mass m moves in the 2-dimensional potential

$$V = \frac{1}{2} k \sin^2 \sqrt{x^2+y^2-xy}.$$

(a) Write the Lagrangian for the system.
(b) Is the origin a stable equilibrium for the particle?
(c) Write the Lagrangian appropriate for small oscillations about $x = y = 0$.
(d) Calculate the normal frequencies.
(e) Write the general small oscillation solution and sketch the normal modes.

Solution:

(a) $L = \frac{1}{2} m(\dot{x}^2+\dot{y}^2) - \frac{1}{2} k \sin^2 \sqrt{x^2+y^2-xy}$

(b) yes

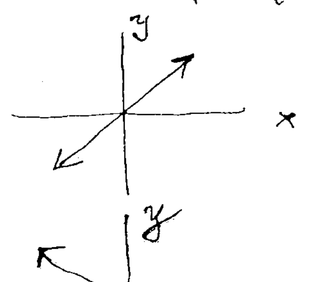
(c) $L = \frac{1}{2} m(\dot{x}^2+\dot{y}^2) - \frac{1}{2} k(x^2+y^2-xy)$

(d)
$$m\ddot{x} = k(x - y/2) \qquad\qquad m(\ddot{x}+\ddot{y}) = -\frac{k}{2}(x+y) \quad (i)$$
$$m\ddot{y} = k(y - x/2) \;\Rightarrow\; m(\ddot{x}-\ddot{y}) = -\frac{3k}{2}(x-y) \quad (ii)$$

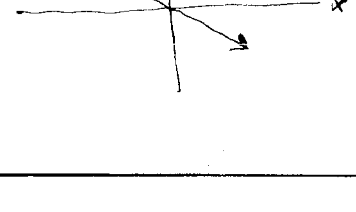
for (i) $\omega_1 = \sqrt{\frac{k}{2m}}$ for (ii) $\omega_2 = \sqrt{\frac{3k}{2m}}$

(e)
$$\begin{pmatrix} x \\ y \end{pmatrix} = A\cos(\omega_1 t+\beta_1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + B\cos(\omega_2 t+\beta_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

ω_1-mode

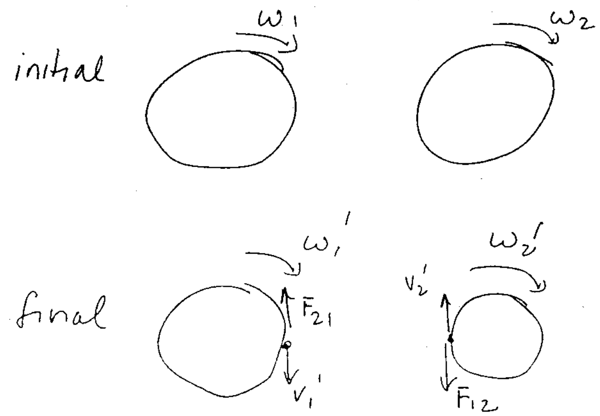


ω_2-mode



Mechanics Solution - Tuts

Thursday, December 18, 2008
7:07 AM



Since the tangential final speeds must be the same (but opposite directions)

$$\omega_1' r_1 = -\omega_2' r_2 \quad (5)$$

$$\Rightarrow \quad \omega_2' = -\frac{r_1}{r_2}\omega_1'$$

Since the integral of the torque equals the change in angular momentum

$$I_1(\omega_1' - \omega_1) = -\int |r_1||F_{21}|dt \quad (5)$$

and $I_2(\omega_2' - \omega_2) = -\int |r_2||F_{12}|dt$ → but $|F_{21}| = |F_{12}|$ from Newtons 3rd Law

$$\therefore \quad \frac{I_1}{r_1}(\omega_1' - \omega_1) = \frac{I_2}{r_2}(\omega_2' - \omega_2)$$

subst in for $\omega_2' = -\frac{r_1}{r_2}\omega_1'$

$$\therefore \quad \frac{I_1}{r_1}\omega_1' + \frac{I_2}{r_2}\cdot\frac{r_1}{r_2}\omega_1' = \frac{I_1}{r_1}\omega_1 - \frac{I_2}{r_2}\omega_2$$

$$\omega_1' = \frac{\frac{I_1}{r_1}\omega_1 - \frac{I_2}{r_2}\omega_2}{\frac{I_1}{r_1} + I_2\frac{r_1}{r_2}} = \frac{\frac{L_1}{r_1} - \frac{L_2}{r_2}}{\frac{I_1}{r_1} + \frac{I_2 r_1}{r_2}}$$

with $I = \frac{1}{2}mr^2$ and some algebra (2)

$$\boxed{\omega_1' = \frac{m_1 r_1 \omega_1 - m_2 r_2 \omega_2}{(m_1 + m_2) r_1}} \quad (3)$$

$$\frac{I_1}{r_1}\omega_1' - \frac{I_1}{r_1}\omega_1 = \frac{I_2}{r_2}(\omega_2' - \omega_2)$$

~~$\frac{I_1}{r_1}\omega_1' + \frac{I_2}{r_2}\omega_2' = I_1$~~