

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 14, 2008
9:00 AM – 11:00 AM

Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 1; Section 1(Classical Mechanics) Question 3, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

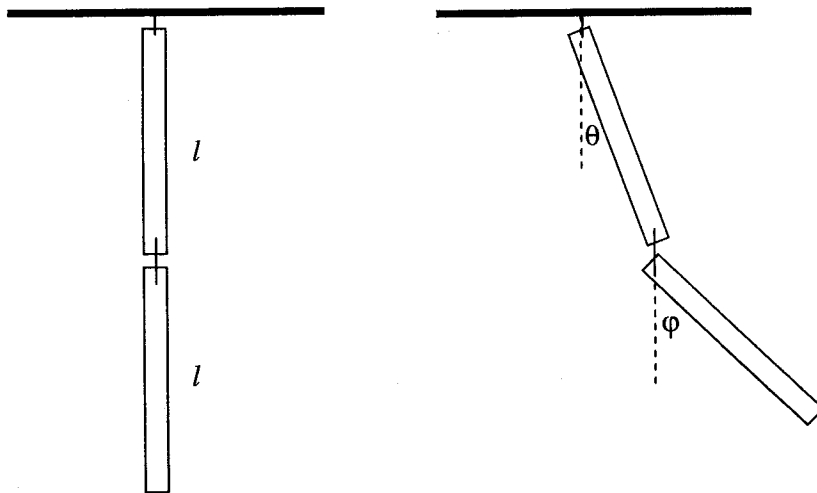
Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is **NOT** permitted.

Questions should be directed to the proctor.

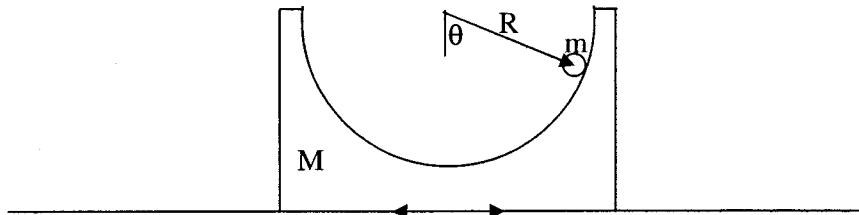
Good luck!!

1. You are mountain climbing on a conical peak described by the equation $z = -\sqrt{x^2 + y^2}$. There is a storm coming and you need to take refuge quickly. What is the equation of the shortest path to the refuge at position $(-1, 0, -1)$ if you are currently located at $(1, 0, -1)$.

2. Two identical rods of mass m and length l are connected to the ceiling and together vertically by small flexible pieces of string. The system then forms a physical double pendulum. Find the frequencies of the normal modes of this system for small oscillations around the equilibrium position. Describe the motion of each of the normal modes.



3. A particle of mass m is constrained to slide without friction on the surface of circular bowl of mass M . The circular bowl has an inner radius R and is free to slide along the horizontal surface without friction. Find the frequency of the normal mode of this system for small oscillations around the equilibrium position at the bottom of the bowl. Describe the motion for this normal mode oscillation.

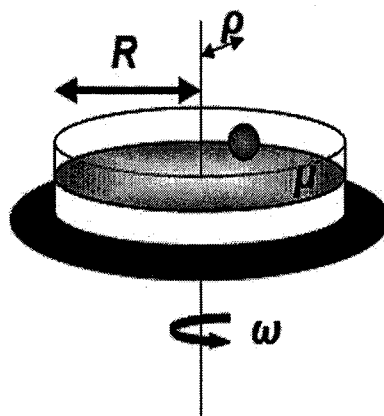


4. A railroad car can move on a frictionless track. The railroad car has a mass M and is initially at rest. In addition, N people (each of mass m) are initially standing at rest on the car.

- (a) Consider the case where all N people run to the end of the railroad car in unison and reach a speed, relative to the car, of v_r . At that point they all jump off at once. Calculate the velocity of the car relative to the ground, after all the people have jumped off.
- (b) Now consider a different case, in which the people jump off one at a time. In other words the people remain at rest relative to the car, while one of them runs to the end, attains a relative speed of v_r and jumps off. Then the next person starts running, attains a relative speed v_r and jumps off. That continues until all N people have jumped off. Find an expression for the final velocity of the railroad car relative to the ground.
- (c) In which case, (a) or (b), does the railroad car attain a greater velocity?

5. We place a cylindrical pan of radius R at the center of a record player. It rotates with a constant angular frequency ω and the pan is half-filled with water.

- (a) Compute the equilibrium surface shape of the fluid, which has mass density μ , taking into account the downward acceleration g due to gravity.
- (b) A ping-pong ball of mass m and radius b is placed on the rotating fluid at a radial position ρ from the rotation axis. It is given an initial tangent velocity $v(0) = \omega\rho$. Compute all the forces on the ball and discuss qualitatively the subsequent motion of the ball.



Quals 2008

Elena Aprile, Fall 2007

Section 1: Classical Mechanics: Mountain Climbing

You are mountain climbing on a conical peak described by the equation $z = -\sqrt{x^2 + y^2}$. There is a storm coming and you need to take refuge quickly. What is the equation of the shortest path to the refuge at position $(-1, 0, -1)$ if you are now located at $(1, 0, -1)$.

Solution

In cylindrical coordinates the length element is $ds = \sqrt{dr^2 + r^2d\phi^2 + dz^2}$ and since the mountain is described by the equation $z = -r$ then the length element on the mountain is given by

$$ds = \sqrt{2dr^2 + r^2d\phi^2}. \quad (1)$$

To find the optimal path we need to minimize the functional

$$\mathcal{P}[r(\phi)] = \int \sqrt{2dr^2 + r^2d\phi^2} = \int d\phi \sqrt{r^2 + 2\dot{r}^2} = \int d\phi \mathcal{L}(r, \dot{r}, \phi). \quad (2)$$

Consequently, the shortest path is the solution of

$$\frac{d}{d\phi} \frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = 0 \quad (3)$$

$$\frac{2\ddot{r}}{\sqrt{r^2 + 2\dot{r}^2}} - \frac{2\dot{r}(\dot{r}r + 2\ddot{r}\dot{r})}{(r^2 + 2\dot{r}^2)^{3/2}} - \frac{r}{\sqrt{r^2 + 2\dot{r}^2}} = 0 \quad (4)$$

$$2\ddot{r}(r^2 + 2\dot{r}^2) - 2\dot{r}(\dot{r}r + 2\ddot{r}\dot{r}) - r(r^2 + 2\dot{r}^2) = 0 \quad (5)$$

$$r^2 + 4\dot{r}^2 - 2\ddot{r}r = 0. \quad (6)$$

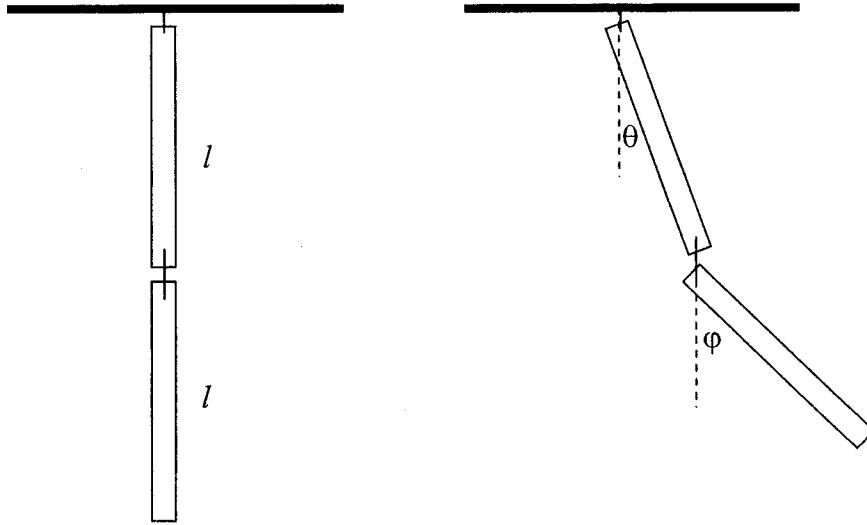
With $r = 1/u$ then $\dot{r} = -\dot{u}/u^2$ and $\ddot{r} = 2\ddot{u}/u^3 - \dot{u}/u^2$ which leads to

$$\frac{1}{u^2} + \frac{4\dot{u}^2}{u^4} - \frac{2}{u} \left(\frac{2\ddot{u}}{u^3} - \frac{\dot{u}}{u^2} \right) = 0, \quad \frac{1}{u^2} + \frac{2\ddot{u}}{u^3} = 0, \quad \ddot{u} + \frac{1}{2}u = 0. \quad (7)$$

The general solution for the optimal path is thus $u(\phi) = A \cos \frac{\phi}{\sqrt{2}} + B \sin \frac{\phi}{\sqrt{2}}$. From the initial position constraint $(r_0, \phi_0) = (1, 0)$, we obtain $u(0) = A$ which leads to $A = 1$. Similarly, from the final position constraint $(r_1, \phi_1) = (1, \pi)$ we obtain $u(\pi) = \cos \frac{\pi}{\sqrt{2}} + B \sin \frac{\pi}{\sqrt{2}}$ giving $B = (1 - \cos \frac{\pi}{\sqrt{2}}) / \sin \frac{\pi}{\sqrt{2}}$. Putting everything together we obtain that the shortest path to the refuge is the one described by the equation

$$r(\phi) = \cos \frac{\pi}{2\sqrt{2}} \sec \frac{\pi - 2\phi}{2\sqrt{2}}. \quad (8)$$

Two identical rods of mass m and length l are connected to the ceiling and together vertically by small flexible pieces of string. The system then forms a physical double pendulum. Find the frequencies of the normal modes of this system for small oscillations around the equilibrium position. Describe the motion of each of the normal modes.



Solution:

Let θ (ϕ) be the angle of the rod with respect to the vertical for the top (bottom) rod.

$$T = \frac{1}{2} \left(m \left(\frac{l}{2} \dot{\theta} \right)^2 + \frac{1}{12} ml^2 \dot{\theta}^2 + m \left(l \dot{\theta} + \frac{l}{2} \dot{\phi} \right)^2 + \frac{1}{12} ml^2 \dot{\phi}^2 \right)$$

$$U = mg \frac{l}{2} (1 - \cos \theta) + mg \left(\frac{3}{2} l - \left(l \cos \theta + \frac{l}{2} \cos \phi \right) \right) \approx mgl \left(\frac{\theta^2}{4} + \left(\frac{\theta^2}{2} + \frac{\phi^2}{4} \right) \right)$$

$$L = T - U = \frac{4}{6} ml^2 \dot{\theta}^2 + \frac{ml^2}{2} \dot{\theta} \dot{\phi} + \frac{1}{6} ml^2 \dot{\phi}^2 - \frac{mgl}{4} (3\theta^2 + \phi^2)$$

Then Lagrange's equations are then given by

$$\frac{1}{2} \left(\frac{8}{3} l \ddot{\theta} + l \ddot{\phi} + \frac{3}{2} g \theta \right) = 0 \quad \frac{1}{2} \left(l \ddot{\theta} + \frac{2}{3} l \ddot{\phi} + \frac{1}{2} g \phi \right) = 0$$

$$\frac{1}{2} \left(\frac{8}{3} \ddot{\theta} + \ddot{\phi} + \frac{3}{2} \omega_0^2 \theta \right) = 0 \quad \frac{1}{2} \left(\ddot{\theta} + \frac{2}{3} \ddot{\phi} + \frac{1}{2} \omega_0^2 \phi \right) = 0 \quad \text{where } \omega_0^2 = \frac{g}{l}$$

Assuming small oscillations with $\theta = A \cos \omega t$ and $\phi = B \cos \omega t$ gives

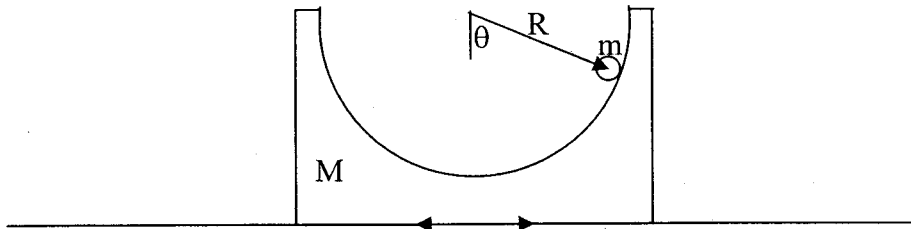
$$\begin{pmatrix} \frac{3}{2} \omega_0^2 - \frac{4}{3} \omega^2 & -\frac{\omega^2}{2} \\ -\frac{\omega^2}{2} & \frac{1}{2} \omega_0^2 - \frac{1}{3} \omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

which yields normal mode frequencies of $\omega^2 = \left(3 \pm \frac{6}{\sqrt{7}} \right) \omega_0^2 = \begin{cases} 5.27 \omega_0^2 \\ 0.73 \omega_0^2 \end{cases}$

For the $\omega^2 = \left(3 + \frac{6}{\sqrt{7}}\right)\omega_0^2$ frequency, $B = \left(\frac{-2\sqrt{7}}{3} - \frac{1}{3}\right)A = -2.10A$

For the $\omega^2 = \left(3 - \frac{6}{\sqrt{7}}\right)\omega_0^2$ frequency, $B = \left(\frac{2\sqrt{7}}{3} - \frac{1}{3}\right)A = -1.43A$

A particle of mass m is constrained to slide without friction on the surface of circular bowl of mass M . The circular bowl has an inner radius R and is free to slide along the horizontal surface without friction. Find the frequency of the normal mode of this system for small oscillations around the equilibrium position at the bottom of the bowl. Describe the motion for this normal mode oscillation.



Solution:

Let X be the coordinate of the bowl along the horizontal axis, and θ be angular position of the mass, m . The positive X and θ directions are opposite. Then

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X} - R\dot{\theta})^2, \quad U = mgR(1 - \cos\theta) = \frac{1}{2}mgR\theta^2$$

$$L = T - U = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X} - R\dot{\theta})^2 - \frac{1}{2}mgR\theta^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = -mR(\ddot{X} - R\ddot{\theta}) + mgR\theta, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) - \frac{\partial L}{\partial X} = (M + m)\ddot{X} - mR\ddot{\theta}$$

With $\theta = A\cos\omega t$ and $X = B\cos\omega t$, we have

$$\begin{pmatrix} mgR - mR^2\omega^2 & mR\omega^2 \\ mR\omega^2 & -(m + M)\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

which gives $\omega = \sqrt{\frac{g(m + M)}{RM}}$ for the normal mode frequency.

For this mode, the mass m goes one way and bowl goes the other

to keep the CM fixed since there are no external forces. $B = \frac{mR}{(m + M)}A$

Quals 2008 – Mechanics Problem – Michael Tuts

1. A railroad car can move on a frictionless track. The railroad car has a mass M and is initially at rest. In addition, N people (each on mass m) are initially standing at rest on the car.
 - A. Consider the case where all N people run to the end of the railroad car in unison and reach a speed, relative to the car, of V_r . At that point they all jump off at once. Calculate the velocity of the car relative to the ground, after all the people have jumped off.
 - B. Now consider a different case, in which the people jump off one at a time. In other words the people remain at rest relative to the car, while one of them runs to the end, attains a relative speed of V_r and jumps off. Then the next person starts running, attains a relative speed v_r and jumps off. That continues until all N people have jumped off. Find an expression for the final velocity of the railroad car relative to the ground.
 - C. In which case (A or B) does the railroad car attain a greater velocity?

Answer:

1A. Let v_{car} be the speed of the railroad car relative to the ground. Since there are no external forces on the person-railroad car system, momentum is conserved. So

$$Mv_{car} + Nm(v_{car} - V_r) = 0$$

$$\therefore v_{car} = \frac{Nm}{M + Nm} V_r$$

1B. Consider the situation when we have n people on the railroad car, and one is about to jump off. Hence this is the transition from n to $n-1$ people on the car. Let v_n be the velocity of the railroad car with the n people on it, the total momentum of the railroad car is:

$$p_n = Mv_n + nmv_n$$

After the n^{th} person has jumped off, the total momentum of the railroad car plus the $n-1$ people and the person that jumped off is

$$p_{n-1} = Mv_{n-1} + m(n-1)v_{n-1} + m(v_{n-1} - V_r)$$

$$p_{n-1} = (M + nm)v_{n-1} - mV_r$$

And since there are no external forces acting between that transition, then

$$p_{n-1} = p_n$$

$$Mv_n + nmv_n = (M + nm)v_{n-1} - mV_r$$

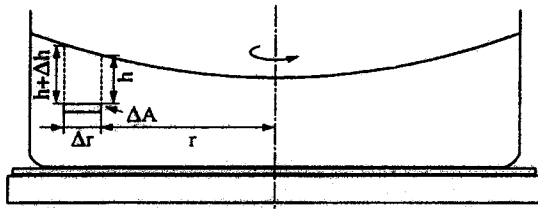
And since $v_N = 0$ because the railroad car and the N people are initially at rest,

$$v_{final} = \sum_{n=1}^N \frac{m}{M + m} V_r$$

1C. Clearly case 1B has the larger final speed.

Due to the internal friction of water and the relatively low angular velocity of the turntable the water will be rotating together with constant angular velocity and there are no turbulence related effects. We can also safely neglect vibration and surface waves/ripples.

- a. The shape of the surface of the liquid will be a paraboloid, centered on the rotation axis. One can see this by applying 'F = ma' to an infinitesimal volume ($\Delta A \Delta h$) of the water:



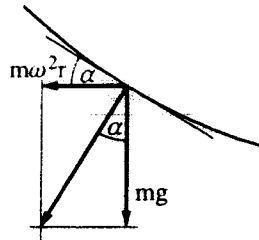
so $[\rho g(h + \Delta h) - \rho g h] \Delta A = \Delta A \cdot \Delta r \cdot \rho \cdot \omega^2 r$

which, after some algebra, gives us (to avoid confusion ρ is not used for the radius here)

$$g \cdot \Delta h = \omega^2 r \cdot \Delta r$$

$$\frac{\Delta h}{\Delta r} = \frac{\omega^2 r}{g}$$

thus the angle between the tangent to the surface and the horizontal:



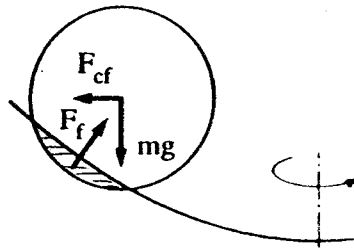
$$\text{tg } \alpha = \frac{\omega^2 r}{g}$$

We can get the shape of the surface through integration:

$$y = \frac{\omega^2}{2g} x^2$$

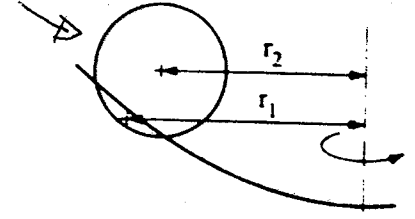
giving us the equation of a paraboloid centered on the rotation axis.

- b. Due to the initial conditions and the friction between the water and the ping-pong ball, the ball will rotate together with the water initially. Neglect the air resistance, which in fact would only speed up the process, but would add significant complications. The forces acting on the ping-pong ball:



Decomposing the forces into horizontal and vertical components, we notice that there is a small net horizontal component:

$$F^H = -|H_f^H| + |H_{cf}^H| \approx -m \omega^2 r_1 + m \omega^2 r_2 < 0 \text{ since } r_2 < r_1$$



due to the paraboloid shaped surface of the liquid. Therefore the ball will slowly spiral into the center and will stay there. (Note, one can also prove that the centrifugal force acting on the ball can be treated as a force concentrated into the center of the ball.)