Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 11, 2006
9:00 AM – 11:00 AM

Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 1; Section 3(QM) Question 5, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single handwritten note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

Problem 1: Section 3 Quantum Mechanics

a) Use only the uncertainty principle to estimate the binding energy E_B of Hydrogen in terms of $m_e, e, \hbar, c$. (Evaluate the answer in terms of Electron Volts to at least 1 digit accuracy, using $m_e c^2 = 5 \times 10^5$ eV and the known value of the fine structure constant, $\alpha = e^2/\hbar c$)

b) In a far off galaxy, long ago, mystery matter changed the Coulomb potential to

$$V(r) = \frac{e^2}{r}\left(\frac{d}{r}\right)^{\epsilon}$$

where d is a new length scale and $|\epsilon| \ll 1$. Assuming that none of the other physical parameters changed, extend part (a) to show that to first order in ϵ, the Bohr radius, r_B, changed to $f\, r_B$ where $f \approx 1 - \epsilon\{1 + \log(d/r_B)\}$.

(Hint: For tiny ϵ, the approximations $1/(1+\epsilon) \approx 1 - \epsilon$ and $x^{\epsilon} \approx 1 + \epsilon \log x$ may be useful)

Problem 2 : Section 3 Quantum Mechanics

Consider *two* identical and non-interacting particles. The particles are in the same spin state, but occupy distinct spatial states a and b, with $\psi_a(x)$ and $\psi_b(x)$ denoting the relevant single-particle 1-D spatial wavefunctions. In the problem below, consider both the case of fermions and bosons.

(a) Write an expression for the normalized two-particle wavefunctions $\psi_F(x_1, x_2)$ and $\psi_B(x_1, x_2)$ for fermions and bosons, respectively. Write the corresponding energy eigenvalues E_F and E_B in terms of the single-particle energies ε_a and ε_b.

(b) Show that the expectation values $\langle x_1^2 \rangle$ and $\langle x_2^2 \rangle$ for the two-particle system satisfy the following relation for both fermions and bosons:

$$\langle x_1^2 \rangle = \langle x_2^2 \rangle = \tfrac{1}{2}\left(\langle x^2 \rangle_a + \langle x^2 \rangle_b\right),$$

where $\langle f(x) \rangle_a \equiv \int_{-\infty}^{+\infty} [\psi_a(x)]^* \, f(x) \, \psi_a(x) \, dx$ (and likewise for b) is the expectation value in the single-particle state.

(c) Define the average separation between the particles as $\langle (x_1 - x_2)^2 \rangle$. Show that the average separation for fermions is always greater than or equal to that for bosons. (For simplicity, you may assume that the single-particle wavefunctions satisfy $\langle x \rangle_a = \langle x \rangle_b$.)

Problem 3 : Section 3 Quantum Mechanics

A particle of mass m moves in a 1-dimensional square well potential

$$V(x) = \begin{cases} 0 & |x| > a \\ -V_0 & -a < x < a \end{cases}$$

1. A plane wave with momentum $\hbar k$ hits the potential well from the left. For certain values of k the wave is perfectly transmitted by the potential. That is, the reflection coefficient vanishes and the transmission coefficient is equal to unity. Determine the values of $E = \hbar^2 k^2/2m$ for which this occurs.

2. The cross section for scattering low energy electrons off xenon atoms exhibits a dip at an electron energy of around 0.7 eV. Suppose the xenon atom can be modeled as a 1-D square well potential. Given that the size of the atom is around 1 Angstrom, estimate the depth of the potential V_0.

Useful facts: $\hbar c \approx 2 \times 10^{-5}\,\text{eV} \cdot \text{cm}$ and $m_e c^2 \approx 500\,\text{keV}$.

Problem 4 : Section 3 Quantum Mechanics

Consider a particle of mass m moving in the following one dimensional potential:

$$V(x) = \begin{cases} \infty & \text{for} \quad 0 < a < x < \infty \\ V_0\, \delta(x) & \text{for} \quad -\infty < x < a \end{cases} \tag{1}$$

where $V_0 a$ is a constant and $a > 0$. Assume that there is a wave, $\exp(+ikx)$ incident on the potential. Write the complete solution in the $x < 0$ region as $u(x) = \exp(+ikx) + R\exp(-ikx)$.

a) Determine $R(k)$ and evaluate its magnitude, $|R|$.

b) Using $R(k)$ determine a transcendental equation for possible bound state energies for $V_0 > 0$ and $V_0 < 0$. (hint: Study the small and large k limits of the equation to set a constraint on $V_0 a$).

c) Sketch qualitatively the functional form of the modulus of the bound state wavefunction(s) in this potential.

Problem 5: Section 3 QM

A particle of charge –e and mass m undergoes simple harmonic motion (spring constant k) in one-dimension. The particle is subject to an electric field of constant value $E = E_o$ along the x-direction. Treating the electrostatic potential as a weak perturbation, determine the ground state energy and the first excited state energy to 2nd order. You may either apply perturbation theory or derive the exact solution to this problem.

Broöijmans

2008

Problem 1: Sec 3 Quantum Mechanics

8

1. a) Use only the uncertainty principle to estimate the binding energy E_B of Hydrogen in terms of $m_e, e, \hbar, c$. (Evaluate the answer in terms of Electron Volts to at least 1 digit accuracy, using $m_e c^2 = 5 \times 10^5$ eV and the known value of the fine structure constant, $\alpha = e^2/\hbar c$)

2. b) In a far off galaxy, mystery matter changes the Coulomb potential to

$$V(r) = \frac{e^2}{r}\left(\frac{d}{r}\right)^{\epsilon}$$

7

where d is a new length scale and $|\epsilon| \ll 1$. Assuming that m_e does not change, show using uncertainty principle that to first order in ϵ, the Bohr radius, r_B, changes to $f \times r_B$ where $f \approx 1 - \epsilon\{1 + \log(d/r_B)\}$.

(Hint: $1/(1+\epsilon) \approx 1 - \epsilon$ and $x^\epsilon \approx 1 + \epsilon \log x$ may be useful)

1 Solution

1. a) $E = p^2/2m - e^2/r > \hbar^2/2mr^2 - e^2/r = E(r)$

   $dE/dr = 0$ gives $r_B = \hbar^2/me^2 = \lambda_e/\alpha$ where $\lambda_e = \hbar/m_e c$ and $\alpha = e^2/\hbar c = 1/137$.
   $E(R_B) = (\hbar^2/2m)m^2e^4/\hbar^2 - e^2me^2/\hbar^2 = -\frac{1}{2}\alpha^2 mc^2 = -1/2(1/137)^2(5 \times 10^5)eV = -5/4 \times 10$ eV, which is reasonably close to the well known Rydberg 13.6 eV.

2. b) Change $E(r,\epsilon) = \hbar^2/2mr^2 - (e^2/r)(d/r)^\epsilon$. Minimize to get $r^{1-\epsilon} = (\hbar^2/me^2(1+\epsilon)d^\epsilon) = r_B/((1+\epsilon)d^\epsilon)$.

   Use $r_B = r_B^{1-\epsilon} r_B^\epsilon$ to write $r/r_B = [(r_B/d)^\epsilon/(1+\epsilon)]^{1/(1-\epsilon)}$.
   Expand $1/(1 \pm \epsilon) \approx 1 \mp \epsilon$. Keep first order only. Use $x^\epsilon \approx 1 + \epsilon \log x$.
   Therefore the new minimum is at $r \approx r_B(1-\epsilon)(r_B/d)^\epsilon = r_B(1 - \epsilon(1 + \log(d/r_B)) = r_B(1 - \epsilon \log(ed/r_B))$.
   $1 + \epsilon \log(r_B/d)$ $-\epsilon$
   For $\epsilon > 0$ the Bohr radius shrinks if $d > r_B/e$.
   Not needed for full credit but for fun: The kinetic $T \approx T_B[1+2\epsilon(1+\log(d/r_B))]$. The pot $V \approx V_B(1+\epsilon\log(ed/r_B))(d/r_B)^\epsilon \approx V_B[1+\epsilon(1+2\log(d/r_B))]$. Recall $V_B = -2T_B = 2E_B$,

   $E = -E_B[1+2\epsilon(1+\log(d/r_B))] + 2E_B[1+\epsilon(1+2\log(d/r_B))] = E_B + \epsilon E_B[1+2\log(d/r_B)]$

QUANTUM MECHANICS PROBLEM (HEINZ)
12/2/05

Consider *two* identical and non-interacting particles. The particles are in the same spin state, but occupy distinct spatial states a and b, with $\psi_a(x)$ and $\psi_b(x)$ denoting the relevant single-particle 1-D spatial wavefunctions. In the below, consider both the case of fermions and bosons.

(a) Write an expression for the normalized two-particle wavefunctions $\psi_F(x_1, x_2)$ and $\psi_B(x_1, x_2)$ for fermions and bosons, respectively. Write the corresponding energy eigenvalues E_F and E_B in terms of the single-particle energies ε_a and ε_b.

(b) Show that the expectation values $\langle x_1^2 \rangle$ and $\langle x_2^2 \rangle$ for the two-particle system satisfy the following relation for both fermions and bosons:

$$\langle x_1^2 \rangle = \langle x_2^2 \rangle = \tfrac{1}{2} (\langle x^2 \rangle_a + \langle x^2 \rangle_b),$$

where $\langle f(x) \rangle_a \equiv \int_{-\infty}^{+\infty} [\psi_a(x)]^* \, f(x) \, \psi_a(x) \, dx$ (and likewise for b) is the expectation value in the single-particle state.

(c) Define the average separation between the particles as $\langle (x_1 - x_2)^2 \rangle$. Show that the average separation for fermions is always greater than or equal to that for bosons. (For simplicity, you may assume that the single-particle wavefunctions satisfy $\langle x \rangle_a = \langle x \rangle_b$.)

HEINZ QM PROBLEM (SOLUTION)

(a) Since the spin of the particles is the same, the spin wavefunction must be symmetric. Thus, the spatial wavefunction must be antisymmetric for fermions and symmetric for bosons.

$\psi_F(x_1,x_2) = [\psi_a(x_1)\psi_b(x_2) - \psi_a(x_2)\psi_b(x_1)]/\sqrt{2}$

$\psi_B(x_1,x_2) = [\quad " \quad + \quad " \quad]/\sqrt{2}$

$E_F = E_B = \epsilon_a + \epsilon_b$ since the particles are non-interacting

(b) $\langle x_1^2 \rangle = \frac{1}{2}\iint dx_1 dx_2 \, |\psi_a(x_1)\psi_b(x_2) \mp \psi_a(x_2)\psi_b(x_1)|^2 x_1^2$

$\langle x_1^2 \rangle = \frac{1}{2}[\langle x^2\rangle_a + \langle x^2\rangle_b]$ using the orthonormality of ψ's

$= \langle x_2^2 \rangle$, analogously.

(c) $\langle (x_1-x_2)^2 \rangle = \langle x_1^2 + x_2^2 \rangle - 2\langle x_1 x_2 \rangle = \langle x^2\rangle_a + \langle x^2\rangle_b - 2\langle x_1 x_2\rangle$

The last term is the one that differs for F and B.

$-2\langle x_1 x_2 \rangle = -\iint dx_1 x_2 \, |\psi_a(x_1)\psi_b(x_2) \mp \psi_a(x_2)\psi_b(x_1)|^2 x_1 x_2$

The direct terms are $\propto \langle x_1 \rangle$ or $\langle x_2 \rangle$ and vanish.
The cross terms are

$= \pm \iint dx_1 dx_2 [\psi_a(x_1)\psi_b(x_2)\psi_a^*(x_2)\psi_b^*(x_1) \, x_1 x_2 + cc$

$= \pm \left(\int dx_1 \, \psi_a(x_1) x_1 \psi_b^*(x_1) x_1\right)\left(\int dx_2 \, \psi_b(x_2) x_2 \psi_a^*(x_2) x_2\right) + cc$

$\equiv \pm[\langle x \rangle_{ab}\langle x \rangle_{ba} + c.c.]$

$= \pm 2|\langle x \rangle_{ab}|^2$, for F, B respectively

∴ Fermions are further apart than bosons. ✓

Sec 3 #3

Dan Kabat
11/15/05

Quantum

A particle of mass m moves in a 1-dimensional square well potential

$$V(x) = \begin{cases} 0 & |x| > a \\ -V_0 & -a < x < a \end{cases}$$

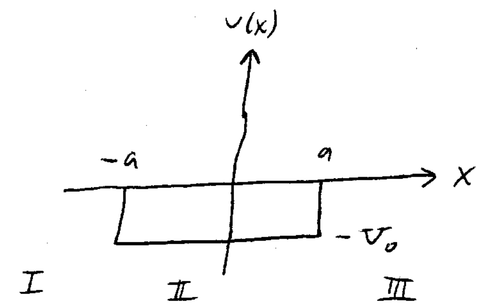
1. A plane wave with momentum $\hbar k$ hits the potential well from the left. For certain values of k the wave is perfectly transmitted by the potential. That is, the reflection coefficient vanishes and the transmission coefficient is equal to unity. Determine the values of $E = \hbar^2 k^2/2m$ for which this occurs.

2. The cross section for scattering low energy electrons off xenon atoms exhibits a dip at an electron energy of around 0.7 eV. Suppose the xenon atom can be modeled as a 1-D square well potential. Given that the size of the atom is around 1 Angstrom, estimate the depth of the potential V_0.

Useful facts: $\hbar c \approx 2 \times 10^{-5}\,\text{eV} \cdot \text{cm}$ and $m_e c^2 \approx 500\,\text{keV}$.

Dan Kabat 11/15/05
Sec 3 #3

Quantum problem solution



$\psi_I = e^{ikx}$ $E = \frac{\hbar^2 k^2}{2m}$

$\psi_{II} = A e^{ik_2 x} + B e^{-ik_2 x}$ $k_2 = \frac{1}{\hbar}\sqrt{2m(E+V_0)}$

$\psi_{III} = C e^{ikx}$

match at $x = -a \Rightarrow$ $e^{-ika} = A e^{-ik_2 a} + B e^{ik_2 a}$

$ik e^{-ika} = ik_2 (A e^{-ik_2 a} - B e^{ik_2 a})$

match at $x = +a \Rightarrow C e^{ika} = A e^{ik_2 a} + B e^{-ik_2 a}$

$ik C e^{ika} = ik_2 (A e^{ik_2 a} - B e^{-ik_2 a})$

Four equations, three unknowns. Get a solution iff

$C = \pm e^{-2ika}$ and $e^{ik_2 a} = \pm 1$ or $\pm i$

$\Rightarrow k_2 = \frac{n\pi}{2a}$ $n = 0, 1, 2, \ldots$

$E = \frac{n^2 \pi^2 \hbar^2}{8ma^2} - V_0$ (need $E > 0$ for scattering state)

The xenon dip presumably corresponds to $n=1$, so

$$V_0 = \frac{\pi^2 \hbar^2}{8ma^2} - E$$

$$= \frac{\pi^2 (2\times 10^{-5}\, eV \cdot cm)^2}{8 \times 500\, keV \times (10^{-8}\, cm)^2} - 0.7\, eV$$

$$\approx 9\, eV$$

Quantum Mechanics Quals 2006

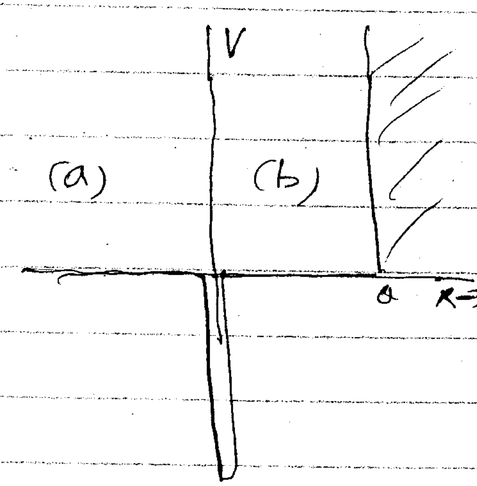
A. Mueller
NOV 21 2005
SEC 3 #4

Consider a particle of mass m moving in a potential $V(x)$ where $V(x)=\infty$ for $x>a$ and $V(x)=V_0\,\delta(x)$ for $-\infty<x<a$ with V_0 a constant. Further suppose there is a wave e^{ikx} incident on the potential. Write $u(x)=e^{ikx}+Re^{-ikx}$ to describe the wavefunction of the particle for $x<0$.

~~(i) Evaluate R. What is |R|.~~

(ii) From R determine the possible bound state energies for $V_0>0$ and for $V_0<0$.

Solution.



Region (a): $u=e^{ikx}+Re^{-ikx}$

Region (b): $u=A\sin k(x-a)$

$u(0+)=u(0-)\Rightarrow 1+R=-A\sin ka$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2}+V(u)=Eu$$

$$\Rightarrow -\frac{\hbar^2}{2m}\underbrace{(u'(0+)-u'(0-))}_{Ak\cos ka-ik(1-R)}+V_0u(0)=0$$

$$\underbrace{Ak\cos ka}_{-\frac{k(1+R)}{\tan ka}}-ik(1-R)=\frac{2m}{\hbar^2}V_0(1+R)$$

gives

$$R=-\left(\frac{k/\tan ka+2mV_0/\hbar^2+ik}{k/\tan ka+2mV_0/\hbar^2-ik}\right) \qquad |R|=1$$

For bound state, $E=-\hbar K^2/2m$ with $k=iK$ and $R(k=iK)=\infty$.

get $K=-mV_0/\hbar^2\,(1-e^{-2Ka})$

$\Rightarrow$ bound state only if $V_0<0$ and $\frac{2mV_0a}{\hbar^2}>0$.

Sec 3 #5

Chuck Hailey's 2006 Quals problem (typed by Elena)
12/5/05

Quantum problem:

A particle of charge –e and mass m undergoes simple harmonic motion (spring constant k) in one-dimension. The particle is subject to an electric field of constant value $E = E_o$ along the x-direction. Treating the electrostatic potential as a weak perturbation, determine the ground state energy and the first excited state energy to 2^{nd} order. If you do not want to apply perturbation theory feel free to seek an exact solution to the problem.

Solution: This is most easily done with operators. Sec3 #5
Hailey QM

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}Kx^2 - eE_0 x = \mathcal{E}\psi$$ ignoring the perturbation, $q=\sqrt{\alpha}\,x$ $\alpha = \frac{m\omega}{\hbar}$ brings the SHO to the form $(p^2+q^2)\psi = \frac{2\mathcal{E}}{\hbar\omega}\psi$ $p = -i\frac{\partial}{\partial q}$

And the canonical transformation $q = \frac{1}{\sqrt{2}}(a^+ + a)$ Allows us to use operators. The perturbation is

$$V_p = -eE_0 x = -\frac{eE_0}{\sqrt{\alpha}}q = -bq \qquad b \equiv \frac{eE_0}{\sqrt{\alpha}}$$

1st order shifts: $\delta E_0 \propto \langle 0|V_p|0\rangle \propto \langle 0|q|0\rangle = 0$ by parity

$\delta E_1 \propto \langle 1|V_p|1\rangle \propto \langle 1|q|1\rangle = 0$ by parity

2) order: $$\delta E_0^{(2)} = \sum_{n\neq 0}\frac{\langle 0|V_p|n\rangle\langle n|V_p|0\rangle}{E_0 - E_n}$$

$$\delta E_1^{(2)} = \sum_{n\neq 1}\frac{\langle 1|V_p|n\rangle\langle n|V_p|1\rangle}{E_1 - E_n}$$

We need $\langle 0|q|n\rangle$ And $\langle 1|q|n\rangle$

$$\langle 0|q|n\rangle = \langle 0|\frac{(a^+ + a)}{\sqrt{2}}|n\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{n+1}\,\delta_{0,n+1} + \sqrt{n}\,\delta_{0,n-1}\right)$$

only $\langle 0|q|1\rangle = \frac{1}{\sqrt{2}}$ is non-vanishing

$$\langle 1|q|n\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{n+1}\,\delta_{1,n+1} + \sqrt{n}\,\delta_{1,n-1}\right)$$

$\langle 1|q|0\rangle = \frac{1}{\sqrt{2}}$ $\qquad \langle 1|q|2\rangle = 1$

$$\delta E_0^{(2)} = b^2\left(\frac{(1/\sqrt{2})^2}{-\hbar\omega}\right) = -\frac{1}{2}\frac{e^2E_0^2}{m\omega^2} = -\frac{1}{2}\frac{e^2E_0^2}{K}$$

$$\delta E_1^{(2)} = b^2\left(\frac{\left(\frac{1}{\sqrt{2}}\right)^2}{\hbar\omega} + \frac{1^2}{-\hbar\omega}\right) = -\frac{1}{2}\frac{e^2E_0^2}{m\omega^2} = -\frac{1}{2}\frac{e^2E_0^2}{K}$$

SC 3 #5

So the shifts are the same and

$$E_0 \simeq \frac{1}{2}\hbar\omega - \frac{1}{2}\frac{e^2E_0^2}{m\omega^2} \; ; \quad E_1 = \frac{3}{2}\hbar\omega - \frac{1}{2}\frac{e^2E_0^2}{m\omega^2}$$

You can do with reg. polynomial $\psi_n(q)$ but this would be tougher. It's easier to solve exactly

Exact soln: using $q = \sqrt{\alpha}\,x$ we can write the Hamiltonian as

$$\left[p^2 + q^2 - \frac{2eE_0 q}{\hbar\omega\sqrt{\alpha}}\right]\psi = \frac{2\varepsilon}{\hbar\omega}\psi$$

call $b = \frac{2eE_0}{\hbar\omega\sqrt{\alpha}}$ $\quad -\frac{d^2}{dq^2}\psi + (q^2 - bq)\psi = \frac{2\varepsilon}{\hbar\omega}\psi$

complete the square $\quad -\psi'' + \left(q - \frac{b}{2}\right)^2 - \frac{b^2}{4} = \frac{2\varepsilon}{\hbar\omega}\psi$

And rescaling $q \to \tilde{q} = q - b/2$

$$-\psi'' + \tilde{q}^2\psi = \left(\frac{2\varepsilon}{\hbar\omega} + \frac{b^2}{4}\right)\psi$$

This is just the SHO with new eigenvalues

$$\frac{2\varepsilon}{\hbar\omega} + \frac{b^2}{4} = 2n+1 \qquad \varepsilon_n = \left(n + \tfrac{1}{2}\right)\hbar\omega - \frac{b^2\hbar\omega}{4\cdot 2}$$

$$\varepsilon_n = \left(n+\tfrac{1}{2}\right)\hbar\omega - \frac{e^2E_0^2}{2m\omega^2} \qquad \text{same as before}$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Wednesday, January 11, 2006
11:10 AM – 1:10 PM

Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

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Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

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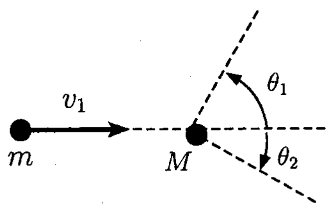
Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

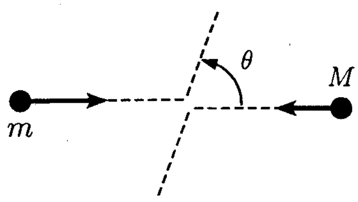
Questions should be directed to the proctor.

Good luck!!

Problem 1: Section 4 Applied QM and Relativity:

A particle of rest mass m and moving with velocity v_1 collides elastically with a stationary particle of rest mass M. Find the recoil and scattering angles in terms of the corresponding angles in the zero momentum system. Verify that your answer has the correct non-relativistic form.





Problem 2: Section 4 Applied QM and Relativity

Quantum Mechanics:
Consider a hydrogen atom in the 1s state. The magnetic interaction of the spin of the proton $\vec{s}_P$ and that of the electron $\vec{s}_e$ is given by the hyperfine Hamiltonian:

$$H_{\mathrm{HF}} = +\frac{8\pi}{3}\frac{g_P g_e}{4 m_P m_e c^2}\,\vec{s}_P \cdot \vec{s}_e\,\delta^3(\vec{r}_e) \tag{1}$$

where $\vec{r}_e$ is the relative coordinate of the electron, g_e and g_P the g-factors for the electron and proton and m_P and m_e their respective masses.

(a) If the hydrogen atom wave function is $\psi(\vec{r}) = e^{-r/a_0}/\sqrt{\pi a_0^3}$ with $a_0 = \hbar^2/(m_e e^2)$, find the splitting between the $F = 0$ and $F = 1$ hyperfine states. (Here $\hbar\vec{F}$ is the total spin of the electron and proton.)

(b) If a weak magnetic field $\vec{B}$ is applied, determine the shift in the energy, $\delta E(B)$, of the lowest hyperfine state.

(c) Compute the magnetic polarizability, $\alpha_B = -\partial^2 \delta E(B)/\partial B^2|_{B=0}$ for this ground state.

Problem 3 : Section 4 Appl. QM and Rel

Two dimensional electron systems can be created on the surface of semiconductors. The electrons are trapped in a potential well and their motion perpendicular to this surface is quantized. The electron system has no degree of freedom perpendicular to the surface (no freedom in the z-direction) but can move freely in the plane (x,y directions).
As an approximation to the well in which the electrons are trapped, we will use a **triangular** potential $V(z) = \mathcal{E}_0 z$ for $z>0$ and V=Infinity for $z<0$. Take $\mathcal{E}_0 = \mathbf{10^5}$ eV/cm.

Part A) Write down the Schrödinger equation for the motion in the z-direction in such a potential well and solve for the wavefunction $\psi_E(z)$, using the Airy function shown in Fig. 1. The Airy function obeys $\frac{d^2}{dw^2}Ai(w) = w\,Ai(w)$ in terms of a variable w and has zeros at approximate values $w_i = -\left[\frac{3}{2}\pi\left(i+\frac{3}{4}\right)\right]^{2/3}$. Discuss the relationship between these values and the energy E in the Schrödinger equation.

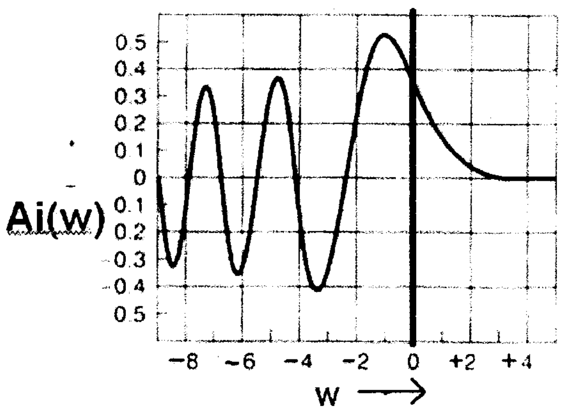


Fig. 1 Airy function

Part B) Find the energy eigenstates $E=E_i$, by inspection of Fig. 1 and by applying the correct boundary conditions. Determine the lowest 3 bound state energies, using the free electron mass, m_e.

Problem 4: Section 4 Applied QM and Relativity

Consider a metal with conduction electron density $\rho \equiv N/V = 5 \times 10^{22}$ per cm^3. Neglect all interactions. The mass of an electron is $m_e c^2 =$ 500 keV, and we assume that the effective mass of electrons in this metal is the same as this "bare electron mass".

a) Describe the ground state of this system

b) What is the characteristic temperature T_c in eV for this metal above which most of the electrons are excited out from the ground state.

c) How do you expect T_c to scale in different metals if ρ and the effective mass m vary? determine the powers a, b with which $T_c \propto \rho^a m^b$

d) Assume next that all electrons combine into N/2 bound pairs with a very large binding energy between two electrons composing the pair. The spin of each pair is zero. We then neglect interaction amoung different pairs. In this simplified situation, describe how the ground state in part (a) would change. Estimate using characteristic quantal and thermo kinetic length scales, the characteristic temperature T_c for this new type of paired electron system.

Problem 5: Section 4 Applied QM and Relativity

The detection of neutrinos from Supernova SN 1987A can be used to put an upper limit on the neutrino mass, m_ν. Show that for two neutrino events with different energies E_1 and E_2, the arrival time difference on Earth can be expressed by a definite function

$$\Delta t = \Delta t(m_\nu, E_1, E_2, L) \tag{1}$$

that depends on the velocity of light c as well as the variables shown.

Calculate an upper limit using typical values $E_1 = 10\,\text{MeV}$, $E_2 = 20\,\text{MeV}$ and the fact that the neutrino pulse from SN 1987A lasted less than 10 s and SN 1987A is $L = 170\,000$ light years away. How does this compare with the current limit (3 eV) from tritium beta decay ?

Amber Miller Sec 4 #1

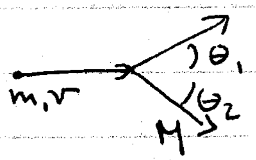
NOV 30 2005

Relativity

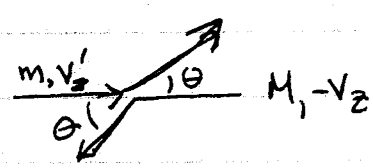
Problem – A particle of rest mass m and velocity v collides elastically with a stationary particle of rest mass M. Find the recoil and scattering angles in terms of the corresponding angles in the zero momentum system. Verify that your answer has the correct non-relativistic form

Solution

Lab frame



zero-p frame



Velocity of zero p system $\quad V_z = \dfrac{\gamma m V}{\gamma m + M}$

$$V' = \frac{V - V_z}{1 - V V_z} = \frac{V - \gamma m V / \gamma m + M}{1 - \frac{V \gamma m V}{\gamma m + M}} = \frac{V(\gamma m + M) - \gamma m V}{\gamma m + M - \gamma m V^2}$$

$$= \frac{MV}{M + \gamma m (1 - V^2)}$$

$$\gamma_z = \frac{1}{\sqrt{1 - V_z^2}}$$

$$V' = \frac{MV}{M + m\sqrt{1 - V^2}}$$

$$\tan\theta_1 = \frac{\sin\theta}{\gamma_z(\cos\theta + V_z/V_1)} \xrightarrow{\text{non-rel}} \frac{\sin\theta}{\cos\theta + m/M}$$

$$\tan\theta_2 = \frac{\sin(\theta + \pi)}{\gamma_z[\cos(\theta + \pi) + V_z/V_z]} = \frac{-\sin\theta}{\gamma_z(1 - \cos\theta)} \longrightarrow \frac{-\sin\theta}{1 - \cos\theta}$$

#4 #2

N. Christ — November 26, 2005

Quals Problems

1. ~~Quantum Mechanics:~~ Applied QM + Relativity

   Consider a hydrogen atom in the 1s state. The magnetic interaction of the spin of the proton $\vec{s}_P$ and that of the electron $\vec{s}_e$ is given by the hyperfine Hamiltonian:

$$H_{\rm HF} = +\frac{8\pi}{3}\frac{g_P g_e}{4 m_P m_e c^2}\,\vec{s}_P \cdot \vec{s}_e\, \delta^3(\vec{r}_e) \qquad (1)$$

   where $\vec{r}_e$ is the relative coordinate of the electron, g_e and g_P the g-factors for the electron and proton and m_P and m_e their respective masses.

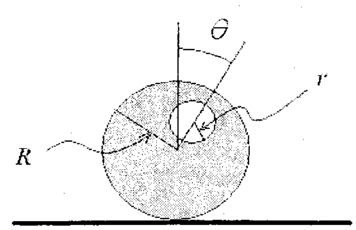
   (a) If the hydrogen atom wave function is $\psi(\vec{r}) = e^{-r/a_0}/\sqrt{\pi a_0^3}$ with $a_0 = \hbar^2/(m_e e^2)$, find the splitting between the $F = 0$ and $F = 1$ hyperfine states. (Here $\hbar\vec{F}$ is the total spin of the electron and proton.) [8 points]

   (b) If a weak magnetic field $\vec{B}$ is applied, determine the shift in the energy, $\delta E(B)$, of the lowest hyperfine state. [10 points]

   (c) Compute the magnetic polarizability, $\alpha_B = -\partial^2\delta E(B)/\partial B^2|_{B=0}$ for this ground state. [2 points]

2. Mechanics:

   A cylinder of length L, radius R and mass density ρ rolls on a horizontal surface without slipping. A hole of radius $r < R$ has been drilled through the cylinder parallel to its axis at a distance $R/2$ from its center. Describe the orientation of the cylinder by specifying the angle θ between the vertical direction and a line connecting the centers of the cylinder and the hole. If initially the cylinder is at rest but θ has a small non-zero value, $\theta(t = 0) = \delta\theta$, describe the subsequent motion. Find the time required for θ to decrease to zero. [20 points]



Suggested Solutions

1. (a) Write the product

$$\vec{s}_P \cdot \vec{s}_e = \frac{1}{2}\left\{(\vec{s}_P + \vec{s}_e)^2 - \vec{s}_P^2 - \vec{s}_e^2)\right\} = \frac{\hbar^2}{2}\left\{f(f+1) - 3/2\right\} \quad (2)$$

where $f = 1$ or 0 for the $F = 1$ and $F = 0$ states. [4 points]
Then simply substitute into the lowest order perturbation theory formula $E_n = \langle n|V|n\rangle$ to determine the ground state as $F = 0$ with hyperfine energy:

$$E_f = \frac{g_P g_e \hbar^2 e^2}{3 m_e m_P c^2 a_0^3}\left\{f(f+1) - \frac{3}{2}\right\} \quad [4 \text{ points}] \qquad (3)$$

(b) For small external field the most important effect will be the mixing of the $f = 0$ and 1 states and the interaction which will do this is

$$\begin{aligned} H_B &= \frac{e}{2c}\left\{-\frac{g_P}{m_P}\vec{s}_P \cdot \vec{B} + \frac{g_e}{m_e}\vec{s}_e \cdot \vec{B}\right\} \\ &\approx \frac{g_e e}{2 m_e c}\vec{s}_e \cdot \vec{B}. \quad [2 \text{ points}] \end{aligned} \qquad (4)$$

We can then use second order perturbation theory to find the shift $\delta E(B)$ in the energy of the $f = 0$ state caused by this term:

$$\begin{aligned} \delta E(B) &= \left(\frac{g_e e B}{2 m_e c}\right)^2 \frac{|\langle f = 1, m_f = 0|s_z|f = 0\rangle|^2}{E_0 - E_1} \quad [4 \text{ points}] \\ &= \frac{3}{16}\frac{g_e}{g_P}\frac{m_P}{m_e}B^2 a_0^3 \quad [4 \text{ points}] \end{aligned} \qquad (5)$$

(c) Differentiating with respect to B then gives:

$$\alpha_B = +\frac{3}{8}\frac{g_e}{g_P}\frac{m_P}{m_e}a_0^3 \quad [2 \text{ points}] \qquad (6)$$

2. Consider rotation about the point of contact, P. Treat the cylinder as a complete cylinder of radius R with mass $M == \rho\pi R^2 L$ and a second of negative mass $-m = -\rho\pi r^2 L$. The first cylinder exerts no torque about P while the second exerts:

$$\tau = -\frac{3R}{2}\rho\pi r^2 L g\theta \quad \text{[5 points]} \tag{7}$$

assuming θ to be small.

The moment of inertia about P is that of the cylinder of radius R minus that of r:

$$I = \frac{1}{2}MR^2 + MR^2 - \frac{1}{2}mr^2 - m(3R/2)^2 \quad \text{[5 points]} \tag{8}$$

where the parallel axis theorem has been used.

Finally we can combine these:

$$I\frac{d^2\theta}{dt^2} = \tau = -\frac{3R}{2}\rho\pi r^2 L g\theta \quad \text{[5 points]} \tag{9}$$

which describes simple harmonic motion with period

$$T = \sqrt{2I/3R\rho\pi r^2 L g} \quad \text{[3 points]} \tag{10}$$

Thus, the cylinder will roll back and forth, executing simple harmonic motion about the equilibrium position $\theta = 0$. It will take $T/4$ time units to first reach $\theta = 0$ [2 points].

Qualifier Question Physics 2005, Stormer, Appl. Quantum Mechanics
11/23/05

Two-Dimensional Electron Systems.

Two dimensional electron systems can be created on the surface of semiconductors. The electrons are trapped in a potential well and their motion perpendicular to this surface is quantized. At low electron densities and at Helium temperatures all electrons can be confined to the lowest bound state while the next state is several kT higher in energy. Under such conditions, the electron system has no degree of freedom perpendicular to the surface (z-direction) but can move freely in the plane (x,y directions). It represents a two-dimensional electron system (2DES), which has shown many interesting physical phenomena. This problem establishes some of the energetics of such systems.

A typical implementation of a 2DES is a Silicon Metal-Oxide-Field-Effect-Transistor (Si MOSFET). It consists of a thick, Si single crystal with a layer of oxide at its surface, followed by a thin layer of metal (see Fig. 1) The oxide acts as an insulator and, assuming that there are already a few electrons at the Si/oxide interface, the whole structure resembles a capacitor. E_F is the Fermi level in the Si.

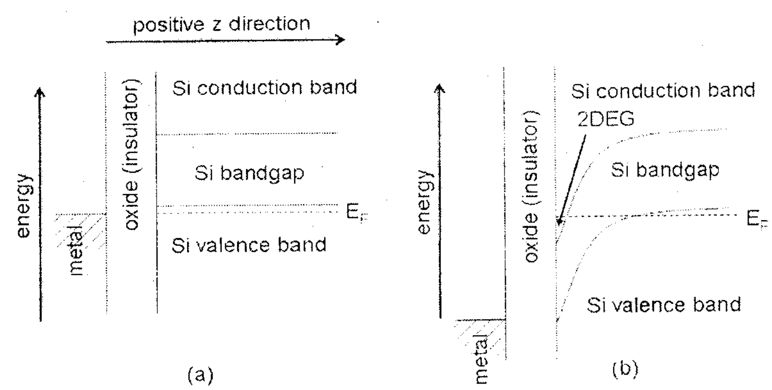


Fig 1. Energetics in a Si MOSFET, before (a) and after (b) biasing

A) Assume the oxide to be d=80nm thick, having a dielectric constant of $\varepsilon_{ox} = 4.5$. Apply 8V bias between the 2DES in the Si and the metal. Calculate the electric field, E_{Si}, within the Si ($\varepsilon_{Si} = 11.8$) right at the interface to the oxide. Neglect any contribution from the 2DES charge density, since quantum mechanically, right at the interface the charge must have dropped to zero.

(2 points)

Sec 4. Prob 3 Solution : Stormer

(8 points)

$$-\frac{\hbar^2}{2m}\psi''(z)+(eE_{Si}z-\varepsilon)\psi(z)=0$$, **replace** $y=z-\frac{\varepsilon}{eE_{Si}}$, $z_0=\frac{\varepsilon}{eE_{Si}}$ **hence**

$$-\frac{\hbar^2}{2m}\psi''(y)+eE_{Si}y\psi(y)=0,$$

hence $\psi''(y)=\frac{2m}{\hbar^2}eE_{Si}y\psi(y)=\beta^3 y\psi(y)$, **with** $\beta^3=\frac{2m}{\hbar^2}eE_{Si}$

replacing $x=\beta y$ **we get** $\psi''(x)=x\psi(x)$, **which is solved by the Airy function of Fig.2.**

with $x=\beta y=\beta(z-\frac{\varepsilon}{eE_{Si}})$ **we find** $x_0=-\beta\frac{\varepsilon}{eE_{Si}}$ **or** $\varepsilon=-\frac{eE_{Si}x_0}{\beta}$

(7 points)

For stationary states the wave function has to have a node at the Si/oxide interface. This requires x_0 to coincide with one of the zeros of the Airy function. Use

$$x_i=-\left[\frac{3}{2}\pi\left(i+\frac{3}{4}\right)\right]^{2/3}$$ **from above to arrive at** $\varepsilon_i=\frac{eE_{Si}}{\beta}\left[\frac{3}{2}\pi\left(i+\frac{3}{4}\right)\right]^{2/3}$.

x_1=2.32, x_2=4.08, x_3=5.52, $\frac{eE_{Si}}{\beta}=\left(\frac{\hbar^2}{2m_e}\right)^{1/3}(eE_{Si})^{2/3}=14meV$

therefore ε_1=31meV, ε_2=57meV, ε_3= 77meV for a free electron mass, m_e.

Stormer
Sec 4 #3 ?

E-field within oxide: E_{ox} =8V/80nm/ε_{ox}=2.22 x 10^7 V/m. Continuity of D at Si/oxide interface yields: $E_{Si}\, \varepsilon_{Si} = E_{ox}\, \varepsilon_{ox}$ or E_{Si}=8.46 x 10^6 V/m.

B) As an approximation to the well in which the electrons are trapped (see Fig. 2(b)), we will use a triangular potential well made from the oxide (infinitely high barrier) and the linearly varying potential due to the E_{Si}, calculated in A). Write down the Schrödinger equation for the motion in the z-direction in such a well and solve it, using the Airy function shown in Fig. 2, with the properties $Ai''(x) = xAi(x)$ and zeros at approximate position $x_i = -\left[\frac{3}{2}\pi\left(i+\frac{3}{4}\right)\right]^{2/3}$ .Note the relationship between position and energy for a ready solution.

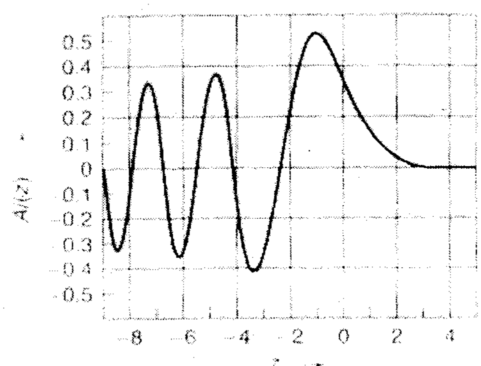


Fig. 2 Airy function

(4 points)

$-\frac{\hbar^2}{2m}\psi''(z) + (eE_{Si}z - \varepsilon)\psi(z) = 0$, **replace** $y = z - \frac{\varepsilon}{eE_{Si}}$, $z_0 = \frac{\varepsilon}{eE_{Si}}$ **hence**

$-\frac{\hbar^2}{2m}\psi''(y) + eE_{Si}\, y\psi(y) = 0$,

hence $\psi''(y) = \frac{2m}{\hbar^2}eE_{Si}\, y\psi(y) = \beta^3 y\psi(y)$, **with** $\beta^3 = \frac{2m}{\hbar^2}eE_{Si}$

replacing $x = \beta y$ **we get** $\psi''(x) = x\psi(y)$, **which is solved by the Airy function of Fig.2.**

with $x = \beta y = \beta(z - \frac{\varepsilon}{eE_{Si}})$ **we find** $x_0 = -\beta\frac{\varepsilon}{eE_{Si}}$ **or** $\varepsilon = -\frac{eE_{Si}x_0}{\beta}$

C) Find the energy eigenstates ε_i , by inspection of Fig. 2 and by applying the correct boundary conditions. Determine the lowest 3 bound state energies, using the free electron mass, m_e.

(3 points)

Sec 4 # 3 ?
Stormer

For stationary states the wave function has to have a node at the Si/oxide interface. This requires x_0 to coincide with one of the zeros of the Airy function. Use

$$x_i = -\left[\frac{3}{2}\pi\left(i+\frac{3}{4}\right)\right]^{2/3}$$ **from above to arrive at** $$\varepsilon_i = \frac{eE_{Si}}{\beta}\left[\frac{3}{2}\pi\left(i+\frac{3}{4}\right)\right]^{2/3}.$$

x_1=2.32, x_2=4.08, x_3=5.52, $\frac{eE_{Si}}{\beta} = \left(\frac{\hbar^2}{2m_e}\right)^{1/3}(eE_{Si})^{2/3} = 14meV$

therefore ε_1=31meV, ε_2=57meV, ε_3= 77meV for a free electron mass, m_e.

D) While you used the free electron mass to arrive at the previous result, the mass in silicon deviates from the free electron mass and is not isotropic. In Si the energy dispersion around the conduction band minimum, appropriate for the above considerations, reads $\varepsilon(\vec{k}) = \frac{\hbar^2}{2}\sum_{\mu\nu} k_\nu (M^{-1})_{\mu\nu} k_\nu$ with k_μ, k_ν being k-vectors, and M being the mass tensor $M = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.9 \end{bmatrix} m_e$ with standard x,y,z notation. What is the effect of this on the previous calculation? By what factor need the eigenstates be scaled?

(2 points)

Only the z-mass m_z=0.9m_e is relevant since the electron motion studied is parallel to z. All energies need to be scaled (increased) by a factor $0.9^{-1/3}$=1.04.

E) To further the description of the 2DES lets assume the capacitor model holds exactly and at V=0 there are negligible carriers in the 2DES, but they are starting to accumulate at V_0=1V (threshould). What is the carrier density at the 8V bias we applied?

(2 point)

Q=ne=C'(V-V_0)=($\varepsilon_{ox}\varepsilon_0$/d)(V-$V_0$), where C' is the capacitance per unit area and n the carrier density per unit area. Hence n=($\varepsilon_{ox}\varepsilon_0$/ed)(V-$V_0$)= 4.5 x 5.5x$10^7$(eV/m)/e80x$10^{-9}$(m) x 7V=2.1x$10^{16}$$m^{-2}$.

F) Using $D(\varepsilon) = 2m/\pi\hbar^2$ for the density of states of this 2DES, calculate the energy to which the 2D states are filled up. Make sure to use the correct mass deduced from the mass tensor in D). Does this filling reach the second energy level calculated in C) or do all electrons "fit" into the lowest energy level?

(2 points)

Sec 4 # 5: Stormer

We need to use the x,y mass (transverse mass) $0.2m_e$. With this $D(\varepsilon)=1.67\text{x}10^{18}$ $(\text{eVm}^2)^{-1}$. Dividing $n=2.1\text{x}10^{16}\text{m}^{-2}$ by $D(\varepsilon)$ yields 12.6meV, which is less than the ε_1 to ε_2 spacing. Therefore only the lowest subband is filled.

Sec 4 # 4

Subject: Re: Quals problem 1: applied quantum Tomo Uemura
From: "Yasutomo J. Uemura" <tomo@lorentz.phys.columbia.edu>
Date: Wed, 30 Nov 2005 12:59:19 -0500 (EST)
To: Lalla Grimes <lalla@phys.columbia.edu>

Dear Quals committee:

If you adopt this problem, do NOT say Fermi Energy nor BE condensation in the problem. These are what students are to supposed to find out.

--- Tomo

On Wed, 30 Nov 2005, Yasutomo J. Uemura wrote:

Possible Quals Problem:

(1) Fermi energy, Bose-Einstein condensation

1-a. We consider a system of spin=1/2 neutral (chargeless) particle without interaction among each other. We consider a 3-dimensional system.
We have n such partciles, and each having the mass m.
a-1. Describe the ground state of this system
a-2. The characteristic temperature T_{a} of this system is proportional to the power of n and m as $T_{a} \propto n^{xa}m^{ya}$ obtain the power xa and ya. (hand-waving argument is enough)
a-3. Obtain the exact form of T_{a}.

1-b. We now consider a system where two of these particles are very strongly coupled to form a composite particle of spin = 0. The number of particle is now n/2, while the mass of the new composite particle is 2m. There is no interaction among the different composite particles.
b-1. Describe the ground state of this new system,
b-2. The characteristic temperature $T_{b} of this system is proportional to the power of n and m as $T_{b} \propto n^{xb}m^{yb}$. Obtain the power xb and yb. Hand-waving argument is sufficient.
b-3. When we compare T_{a} and T_{b}, which is higher ? Describe the reasonings.

to the committee: if you think this is too easy, then we can add

b-4. Obtain the exact value of T_{b}. --- this is not easy,

Sincerely yours,

Tomo Uemura

Tomo Uemura 1&2

Quals. Prob. 4. Section 4. Relativity SOLUTION JAN 17 2008

Fermi Energy, BEC.

a). Fermions will occupy states up to the Fermi Energy in the ground state.

b). Fermi Temperature T_F

Periodic boundary condition $L^3 = V$.

$k = 2\pi n / L$

One state of k per every $\left(\frac{2\pi}{L}\right)^3$

$$N = \frac{4}{3}\pi k_F^3 \times \underset{\uparrow \text{ spin } \downarrow\uparrow}{2} \times \frac{V}{8\pi^3}$$

$$k_F^3 = 3\pi^2 N/V$$

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(3\pi^2 N/V\right)^{\frac{2}{3}}$$

$$= k_B T_F$$

For $N/V = 5 \times 10^{22}$, $m_e c^2 = 511$ keV

$$T_F = 5.05 \text{ eV} \simeq 59{,}000 \text{ °K}$$

c) $$T_F \propto \left(\frac{N}{V}\right)^{2/3} \cdot (m_e)^{-1}$$

d) making bosons $N/2$ $\equiv n_b$ with mass $2m_e \equiv m_b$

ground state : Bose Einstein Condensation.

BEC occurs when thermal wavelength λ becomes comparable to interboson distance

$$\frac{3}{2} k_B T_B = \frac{\hbar^2 k^2}{2 m_b} \qquad k = \frac{2\pi}{\lambda} \cong 2\pi \left(\frac{n_b}{V}\right)^{1/3}$$

$$k_B T_B \sim \frac{1}{3} \cdot \frac{h^2}{2 m_e} \left(\frac{n_e}{V}\right)^{2/3} \cdot \left(\frac{1}{2}\right)^{2/3}$$

$$\sim \frac{\hbar^2}{2 m_e} \cdot \left(\frac{N}{V}\right)^{2/3} \cdot (8.27)$$

$\longrightarrow$ slightly smaller than $k_B T_F$

Do it rigorously

$$k_B T_{BEC} = (2.612)^{-\frac{2}{3}} \left(\frac{N_b}{V}\right)^{\frac{2}{3}} \cdot \left(\frac{\hbar^2}{m_b}\right) \cdot 2\pi$$

$$= (2.612)^{-\frac{2}{3}} \cdot 2\pi \cdot (1.56)^{-1} \cdot \left(\frac{N_e}{V}\right)^{2/3} \times \left(\frac{\hbar^2}{2 m_e}\right)$$

$$= \left(\frac{\hbar^2}{2 m_e}\right) \cdot (2.1) \left(\frac{N_e}{V}\right)^{2/3} \sim \frac{1}{4} k_B T_F$$

Westerhoff
11/28/05
General

sec 5 #6

Problem 1

The acceleration due to gravity on the surface of Mercury is $3.5\,\mathrm{m\,s^{-2}}$. The radius of Mercury is $2.4 \times 10^6\,\mathrm{m}$. Suppose that the atmosphere of Mercury were pure H_2 gas.

(a) What would the temperature be so that the rms speed of the H_2 molecules matched the escape speed ? Qualitatively, what is the effect on the temperature of the remaining gas ?

(b) Would there be a similar effect if the actual temperature was less than the result in (a) ?

(c) If Mercury's atmosphere had two or more components, what would happen to the composition as a function of time ?

Problem 2 (10 points)

sec 4 #5

The detection of neutrinos from Supernova SN 1987A can be used to put an upper limit on the neutrino mass. Show that for two neutrino events with different energies E_1 and E_2, the arrival time difference on Earth is given by

$$\Delta t \simeq \left(\frac{Lm^2c^4}{2c}\right)\left(\frac{1}{E_1^2} - \frac{1}{E_2^2}\right) ,$$

where L is the distance to the supernova, and m is the neutrino mass. Calculate an upper limit using typical values $E_1 = 10\,\mathrm{MeV}$, $E_2 = 20\,\mathrm{MeV}$ and the fact that the neutrino pulse from SN 1987A lasted less than 10 s and SN 1987A is 170 000 light years away. Can this limit compete with current limits from tritium beta decay ?

Problem 2

WESTERHOFF
11/28/05
General # 2

Sec 4
5

For event with energy ϵ, $t_\epsilon = t_{SN} + \frac{L}{\beta}$

Get β using $\epsilon = \frac{m}{\sqrt{1-\beta^2}} \Rightarrow \beta^2 = 1 - \frac{m^2}{\epsilon^2}$,

$$\frac{1}{\beta} \simeq 1 + \frac{m^2}{2\epsilon^2}$$

So

$$t_\epsilon = t_{SN} + L\left(1 + \frac{m^2}{2\epsilon^2}\right)$$

and

$$\Delta t = t_1 - t_2 = \frac{L m^2}{2}\left(\frac{1}{\epsilon_1^2} - \frac{1}{\epsilon_2^2}\right)$$

For the upper limit, $m^2 = \frac{2\Delta t}{L} \frac{1}{1/\epsilon_1^2 - 1/\epsilon_2^2}$,

which gives for $\Delta t < 10$ s

$$m < 22 \text{ eV}.$$

Currently, tritium beta decay gives 3 eV (PDG 2004).