Columbia University
Department of Physics
QUALIFYING EXAMINATION
Thursday, January 13, 2005
9:00 AM – 11:00 AM

General Physics (Part I)
Section 5. Thermodynamics and Statistical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing additional problems). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2; Section 5(General Physics) Question 7, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single note sheet on 8 ½ x 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

Section 5 – Question 1

A quantum gas of N independent, 3D, spin-1, anisotropic oscillators have a Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2}\left(w_1^2 x^2 + w_2^2 y^2 + w_3^2 z^2\right) - \vec{\mu} \cdot \vec{B}$$

where μ is the magnetic dipole moment and B is a uniform magnetic field in the z-direction.

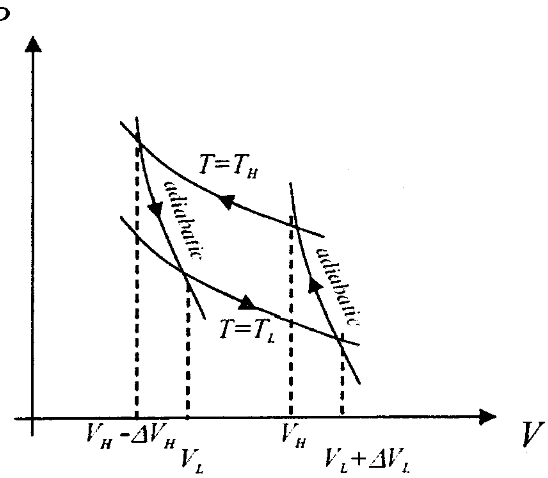
a) Compute the specific heat as a function of temperature.

b) Sketch the temperature dependence of the specific heat for the case

$$\hbar w_1 << \vec{\mu} \cdot \vec{B} << \hbar w_2 << \hbar w_3$$

in the range $0 < T < 2\,\hbar\, w_3$

Section 5 – Question 2



A Carnot engine that consists of a mono-atomic ideal gas with particle number N_c is operating in a refrigerator mode in between two thermal reservoirs at temperature T_H and T_L ($T_H > T_L$) as shown in the accompanying pressure-volume diagram. The engine first makes an isothermal compression of volume from V_H to $V_H - \Delta V_H$ at temperature T_H, followed by the adiabatic volume expansion from $V_H - \Delta V_H$ to V_L. Then the engine undergoes an isothermal expansion from V_L to $V_L + \Delta V_L$ with the temperature held at T_L, followed by an adiabatic compression from $V_L + \Delta V_L$ to V_H, which completes a cycle of refrigeration.

a) Show that: $\dfrac{V_L}{V_H - \Delta V_H} = \dfrac{V_L + \Delta V_L}{V_H} = \left(\dfrac{T_H}{T_L}\right)^{\frac{3}{2}}$.

b) Express the total work applied to this Carnot engine during a cycle of refrigeration in terms of N_c, T_H, T_L, V_L and ΔV_L only. Note that the work done by the engine during two adiabatic processes cancels each other.

c) Find the amount of heat removed from the thermal reservoir at T_L during a cycle.

d) Now, let us assume that the lower temperature thermal reservoir consists of an ideal gas with the particle number N_L. Since the size of this reservoir is finite, the temperature T_L is decreasing every refrigeration cycle. Assuming $\dfrac{N_c}{N_L}\dfrac{\Delta V_L}{V_L} << 1$, show that the temperature of the lower thermal bath after m Carnot cycles is given by: $T_L(m) = T_L(0)e^{-\alpha m}$, where $\alpha = \dfrac{2}{3}\dfrac{N_c}{N_L}\dfrac{\Delta V_L}{V_L}$.

Section 5 – Question 3

A system of N non-interacting, spin ½ particles is described by the Hamiltonian:

$$H = -\mu B \sum_{i=1}^{N} \sigma_i^z$$

where σ_i^z is the z-Pauli matrix associated with particle i:

$$\sigma_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) Calculate the internal energy of the system when it is in thermal equilibrium at field $B_1 > 0$ and temperature T_1.

b) Calculate the change in internal energy (with respect to part a) when the system is maintained in thermal equilibrium at temperature T_1 and the field is decreased to a value B_2 such that $B_1 > B_2 > 0$.

c) Calculate the change in internal energy (with respect to part b) when the system is thermally isolated (decoupled from the reservoir) and the field is increased back to B_1.

d) Calculate the change in internal energy (with respect to part c) when the system is reconnected to the thermal reservoir and allowed to re-equilibrate.

Section 5 – Question 4

A mono-atomic gas is described by the equation of state $P(V - bn) = nRT$.

a) What is the maximum density to which the gas can be compressed ?

b) What is the physics interpretation of the constant b and what property of the atoms can we infer from b ? Give an expression for this property in terms of b and other relevant constants.

c) Suppose the gas has a pressure P_i at volume $V_i = R_i bn$, with $R_i >> 1$. Now, suppose the gas is compressed isothermally to a volume $V_f = 2bn$. Estimate the additional amount of worked needed to compress this gas compared to a gas satisfying the usual ideal gas equation of state. Express your answer in terms of the given constants and any other relevant numerical expressions.

d) Now, suppose the gas is expanded adiabatically back to its original volume, V_i. Calculate the ratio of the temperature for this non-ideal gas to that we would obtain from an ideal gas under the same circumstances.

Section 5 – Question 5

Consider a closed system of N spins in a magnetic field, B. Suppose $N/2+s$ of the spins are "up", $N/2-s$ of the spins are "down" and that the magnetic energy is:

$$U = -2\, m\, B\, s.$$

a) Give the entropy of the system as a function of U.

b) Give the temperature of the system as a function of N and U.

You may wish to use

$$N! \cong \sqrt{2\pi N} \cdot e^{N \ln N - N}$$

and the fact that the total number of states in the configuration above is

$$g(N,s) = \frac{N!}{(N/2+s)!\,(N/2-s)!}$$

Section 5 – Question 6

Consider an ideal gas composed of N He^3 atoms contained in a vessel of volume V. He^3 isotopes have two protons and one neutron in their nuclei. He^3 atoms have spin ½. Describe the kinetic energy states of the He^3 atoms as quantization in a box for a cube of volume $V = L^3$, where L is the length of the cube. Assume, for simplicity, that the system remains gaseous at all temperatures.

Consider the limit of absolute zero of temperature ($T = 0$).

a) What is the difference between the lowest and highest energy states of single He^3 atoms.

b) Obtain an expression for the total kinetic energy as function of N and V.

c) What would be the total kinetic energy for a gas composed of He^4 atoms?

Now assume that the temperature is raised slightly so that T remains small.

d) The kinetic energy of the He^3 ideal gas is written as $U(T) = U(0) + \Delta U(T)$. Show by qualitative considerations that the leading term in $\Delta U(T)$ is proportional to T^2.

2005 Section V general part I Thermodynamics & Statistical mechanics V-1

Gyulassy

Section 5 – Question 1

no solution found

A quantum gas of N independent, 3D, spin-1, anisotropic oscillators have a Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2}\left(w_1^2 x^2 + w_2^2 y^2 + w_3^2 z^2\right) - \vec{\mu} \cdot \vec{B}$$

where μ is the magnetic dipole moment and B is a uniform magnetic field in the z-direction.

a) Compute the specific heat as a function of temperature.

b) Sketch the temperature dependence of the specific heat for the case

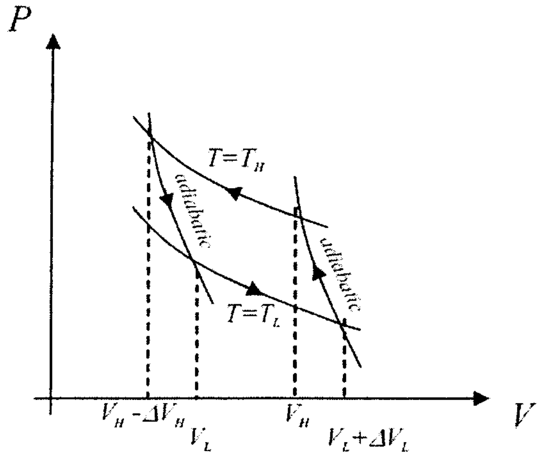
$$\hbar w_1 << \vec{\mu} \cdot \vec{B} << \hbar w_2 << \hbar w_3$$

in the range $0 < T < 2\,\hbar\, w_3$.

Philip Kim   V-2

Thermodynamics (heat engine):

A Carnot engine that consists of a monantomic ideal gas with the particle number N_c is operating in a refrigerator mode in between two thermal reservoirs at temperature T_H and T_L ($T_H > T_L$) as shown in the pressure-volume diagram shown in the below. The engine first makes an isothermal compression of volume from V_H to $V_H - \Delta V_H$ at temperature T_H, followed by the adiabatic volume expansion from $V_H - \Delta V_H$ to V_L. Then the engine undergoes an isothermal expansion from V_L to $V_L + \Delta V_L$ with the temperature holds at T_L, followed by an adiabatic compression from $V_L + \Delta V_L$ to V_H, which completes a cycle of refrigeration.



(a) Show that

$$\frac{V_L}{V_H - \Delta V_H} = \frac{V_L + \Delta V_L}{V_H} = \left(\frac{T_H}{T_L}\right)^{\frac{3}{2}}.$$

(b) Express the total work applied to this Carnot engine during a cycle of refrigeration in terms of N_c, T_H, T_L, V_L and ΔV_L only. Note that the work done by the engine during two adiabatic processes cancels each other.

(c) Find the amount of heat removed from the thermal reservoir at T_L during a cycle.

(d) Now, let us assume that the lower temperature thermal reservoir consists of an ideal gas with the particle number N_L. Since the size of this reservoir is finite, the temperature T_L is decreasing every refrigeration cycle. Assuming $\frac{N_c}{N_L}\frac{\Delta V_L}{V_L} << 1$, show that the temperature of the lower thermal bath after m_{th} Carnot cycles is given by:

$T_L(m) = T_L(0)\, e^{-\alpha m}$, where $\alpha = \frac{2}{3}\frac{N_c}{N_L}\frac{\Delta V_L}{V_L}$.

Philip Kim

Thermo

(a) $dQ = PdV + dU$

For monatomic ideal gas, $\left\{ \begin{array}{l} PV = Nk_BT \\ U = \frac{3}{2}k_BNT \end{array} \right\}$.

During adiabatic processes, $dQ = 0$.

Thus $0 = PdV + dU \Rightarrow 0 = \frac{Nk_BT}{V}dV + \frac{3}{2}k_BN\,dT$

or $\boxed{\frac{dV}{V} = -\frac{3}{2}\frac{dT}{T}} \Rightarrow \ln\left[\frac{V_L}{V_H - \Delta V_H}\right] = \ln\left[\frac{V_L + \Delta V_L}{V_H}\right]$

$= -\frac{3}{2}\ln\left[\frac{T_L}{T_H}\right]$

or $$\boxed{\frac{V_L}{V_H - \Delta V_H} = \frac{V_L + \Delta V_L}{V_H} = \left(\frac{T_H}{T_L}\right)^{3/2}}$$

(b) Total work done by the engine,

$$W_{tot} = W^{adiabatic}_{T_H \to T_L} + W_{T=T_L} + W^{adiabatic}_{T_L \to T_H} + W_{T=T_H}$$

Here, during adiabatic process, $\cancel{dQ} \overset{0}{=} dW + dU$

$$dW^{adiab} = -dU = \frac{3}{2}k_BN_c\,dT$$

Thus $$W^{adiab}_{T_H \to T_L} + W^{adiab}_{T_L \to T_H} = \frac{3}{2}k_BN_c\overbrace{\{(T_L - T_H) + (T_H - T_L)\}}^{0}$$

or $$\boxed{W_{tot} = W_{T=T_L} + W_{T=T_H}}$$

Here during isothermal process

$$dW = P\,dV = \frac{N_c k_B T}{V}\,dV$$

Thus $$W_{T=T_L} = N_c k_B T_L \ln\left(\frac{V_L+\Delta V_L}{V_L}\right)$$

and $$W_{T=T_H} = N_c k_B T_H \ln\left(\frac{V_H-\Delta V_H}{V_H}\right).$$

From the result of (a), we know

$$\frac{V_H-\Delta V_H}{V_H} = \frac{V_L}{V_L+\Delta V_L}$$

Therefore,

$$W_{tot} = N_c k_B T_L \ln\left(\frac{V_L+\Delta V_L}{V_L}\right) + N_c k_B T_H \ln\left(\frac{V_L}{V_L+\Delta V_L}\right)$$

or

$$\left(\begin{array}{c}\text{the total work}\\ \text{applied to the}\\ \text{engine}\end{array}\right) = -W_{tot} = N_c k_B (T_H - T_L)\ln\left(\frac{V_L+\Delta V_L}{V_L}\right)$$

(c) $$dQ = dW + dU \quad (dU \to 0 \text{ for isothermal process}).$$

Thus $$\left(\begin{array}{c}\text{the heat removed}\\ \text{from } T_L \text{ reservoir}\end{array}\right) = -W_{T=T_L}$$

$$= N_c k_B T_L \ln\left(\frac{V_L+\Delta V_L}{V_L}\right)$$

(d) Since the lower reservoir is an ideal gas

$$(-\Delta Q_L) = C_V \Delta T_L = \frac{3}{2} k_B N_L \Delta T_L$$

heat removed from the reservoir

From the result in (c),

$$-N_c k_B T_L \ln\left(\frac{V_L + \Delta V_L}{V_L}\right) = \frac{3}{2} k_B N_L \Delta T_L$$

or

$$\frac{\Delta T_L}{T_L} = -\frac{2}{3}\frac{N_c}{N_L}\ln\left[1 + \frac{\Delta V_L}{V_L}\right] \simeq -\frac{2}{3}\frac{N_c}{N_L}\frac{\Delta V_L}{V_L}$$

per cycle.

Therefore

$$\ln\left[\frac{T_L(m)}{T_L(0)}\right] = -\frac{2}{3}\frac{N_c}{N_L}\frac{\Delta V_L}{V_L} m$$

after m-th cycle.

or

$$T_L(m) = T_L(0)\, e^{-\alpha \cdot m}$$

where $\alpha = \frac{2}{3}\frac{N_c}{N_L}\frac{\Delta V_L}{V_L}$ //

Solution (Revised) by Igor Aleiner

$$F = -NT \ln 2\cosh\frac{\mu B}{T},$$

$$S = N\left[\ln 2\cosh\frac{\mu B}{T} - \frac{\mu B}{T}\tanh\frac{\mu B}{T}\right];$$

$$E = -\left[\mu B \tanh\frac{\mu B}{T}\right]N;$$

Step. 2.

$$\Delta E_2 = N\left[-\mu B_2 \tanh\frac{\mu B_2}{T} + \mu B_1 \tanh\frac{\mu B_1}{T}\right];$$

Step. 3;

$S = $ const; thus $\frac{B_2}{T} = \frac{B_1}{T_1}$; (*)

$$\Delta E_3 = N\left[-\mu B_1 \tanh\frac{\mu B_1}{T_1} + \mu B_2 \tanh\frac{\mu B_2}{T}\right]; \text{ or}$$

using (*)

$$\Delta E_3 = N\left[-\mu B_1 \tanh\frac{\mu B_2}{T} + \mu B_2 \tanh\frac{\mu B_2}{T}\right] =$$

$$= \mu N \tanh\frac{\mu B_2}{T}(B_2 - B_1);$$

Step 4:

$$\Delta E_4 = N\left[-\mu B_1 \tanh\frac{\mu B_1}{T} + \mu B_1 \tanh\frac{\mu B_1}{T_1}\right]$$

$$= N\mu B_1\left[\tanh\frac{\mu B_2}{T} - \tanh\frac{\mu B_1}{T}\right],$$

Thermo: Mills

A system of N noninteracting $S=\frac{1}{2}$ ~~particles~~ spins is described by the Hamiltonian

$$H = -\gamma B \sum_{i=1}^{N} \sigma_i^z$$

where $\sigma_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the z- Pauli matrix associated with particle i.

~~Consi~~

Please determine the work done by the external magnetic field during the following cycle:

(1) System is in thermal equilibrium at field $B_1 > 0$ and temperature T_1

(2) While system is maintained in thermal equilibrium at temperature T_1 the field is decreased to a value B_2 such that $B_1 > B_2 > 0$

(3) The system is thermally isolated (decoupled from reservoir), and the field is increased back to B_1

(4) The system is reconnected to the thermal reservoir, and allowed to re-equilibrate

You are to compute the change in internal energy at each step of this process

Free energy, F, per spin

$$F = -T \ln\left[2\, ch \frac{\mu B}{T}\right]$$

$\Rightarrow$ ~~energy~~ entropy $S = -\frac{\partial F}{\partial T} = \ln 2\, ch \frac{\mu B}{T} - \frac{2\mu B}{T} th \frac{\mu B}{T}$

energy $E = F + TS = -\frac{2\mu B}{T} th \frac{\mu B}{T}$

In step 2, T constant

$$\Delta E_2 = -\frac{2\mu B_1}{T} th \frac{\mu B_1}{T} + \frac{2\mu B_2}{T} th \frac{\mu B_2}{T}$$

In step 3, entropy constant $\Rightarrow$ B/T const

$\Rightarrow T_3 = TB_1/B_2$

$$\Rightarrow E_3 = -\frac{2\mu B_1}{T_3} th \frac{B_1}{T_3} = -\frac{2\mu B_2}{T} th \frac{\mu B_2}{T}$$

$\Delta E = 0$

In step 4, non adiabatic process

$$E_4 = -\frac{2\mu B_1}{T} th \frac{\mu B_1}{T} \Rightarrow \Delta E_4 = -\Delta E_2$$

V-4
No solution provided

Brian Cole
12/17/04
Thermodynamics – Canonical Ensembles (?)

Non-ideal gas
A mono-atomic gas is described by the equation of state $P(V - bn) = nRT$.

a) What is the maximum density to which the gas can be compressed ?
b) What is the physics interpretation of the constant b and what property of the atoms can we infer from b ? Give an expression for this property in terms of b and other relevant constants.
c) Suppose the gas has a pressure P_i at volume $V_i = R_i bn$ with $R_i >> 1$. Now, suppose the gas is compressed isothermally to a volume $V_f = 2bn$. Estimate the additional amount of worked needed to compress this gas compared to a gas satisfying the usual ideal gas equation of state. Express your answer in terms of the given constants and any other relevant numerical expressions.
d) Now, suppose the gas is expanded adiabatically back to its original volume V_i. Calculate the ratio of the temperature for this non-ideal gas to that we would obtain from an ideal gas under the same circumstances.

Cole

(with solution)

Brian Cole
12/17/04
Thermodynamics – Canonical Ensembles (?)

Non-ideal gas
A mono-atomic gas is described by the equation of state $P(V - bn) = nRT$.

2 3 a) What is the maximum density to which the gas can be compressed ?

3 3 b) What is the physics interpretation of the constant b and what property of the atoms can we infer from b ? Give an expression for this property in terms of b and other relevant constants.

5 c) Suppose the gas has a pressure P_i at volume $V_i = R_i bn$ with $R_i >> 1$. Now, suppose the gas is compressed isothermally to a volume $V_f = 2bn$. Estimate the additional amount of worked needed to compress this gas compared to a gas satisfying the usual ideal gas equation of state. Express your answer in terms of the given constants and any other relevant numerical expressions.

5 8 d) Now, suppose the gas is expanded adiabatically back to its original volume V_i. Calculate the ratio of the temperature for this non-ideal gas to that we would obtain from an ideal gas under the same circumstances.

Qualifying Exam 2005 — Section V, problem 4

Brian Cole — Solutions

a) Observe: if $bn > V$, then either P or nRT must be negative → impossible

Thus, $n/V\ \text{max} = 1/b$ → maximum molar density

b) bn has units of volume. Clearly when $bn = V$ the system can no longer be compressed.

Therefore ~~bn~~ b/N_A is an "excluded volume" or "hard core" volume of each atom.

→ Can infer atomic "radius" from $R \sim$ ~~$b^{1/3}$~~ $(b/N_A)^{1/3}$
or more precisely $R \approx$ ~~$\left(\frac{3b}{4\pi}\right)^{1/3}$~~ $\left(\frac{3b}{4\pi N_A}\right)^{1/3}$

c) For isothermal compression, T constant.
~~Word~~ Work done on system: $\Delta W = -\int_{V_i}^{V_f} P(V)\,dV$

So, for given equation of state:

$$\Delta W = -\int_{R_i bn}^{2bn} dV \frac{nRT}{V-bn} = -nRT \ln(V-bn)\Big|_{R_i bn}^{2bn}$$

$$= nRT\ln(R_i) \quad (\text{using } R_i - 1 \approx R_i$$

For ideal gas: $\Delta W = -\int_{R_i bn}^{2bn} dV \frac{nRT}{V} = -nRT \ln V\Big|_{R_i bn}^{2bn}$

Then $\Delta W^{ideal} = nRT \ln(R_i/2)$

With the result: $\Delta W - \Delta W_{ideal} = nRT(\ln R_i - \ln(R_i/2))$
$= nRT \ln 2$

d) The initial temperature before compression is obtained from $P_i(V_i - bn) = nRT_i$.
If $R_i \gg 1$ then $T_i = \frac{P_i b n R_i}{nR} = \frac{P_i b R_i}{R}$

After compression, the temperature is the same.

Then for adiabatic expansion: $dU = -p\,dV$

$\to \frac{3}{2} nR\, dT = -\frac{nRT}{V - bn} dV$

So $\ln T^{3/2} \Big|_{T_i}^{T_f} = -\ln(V - bn)\Big|_{V_i}^{V_f} \Rightarrow T^{3/2}(V - bn) = \text{Const.}$

~~So if $P_i' = \frac{nRT_i'}{2bn - bn} = \frac{RT_i'}{b}$ primes for start of the adiabatic exp.~~

Then: $T_i^{3/2} bn = T_f^{3/2}(R_i - 1)bn \Rightarrow T_f = T_i R_i^{-2/3}$

For ideal gas: $T^{3/2} V = \text{Const.}$ $T_i = \frac{P_i b n R_i}{nR} = \frac{P_i b R_i}{R}$

$\to$ Same $(\approx)$ as for our non-ideal gas

Then $T_i^{3/2}(b n 2) = T_f^{3/2}(R_i b n) \Rightarrow T_f = T_i \left(\frac{2}{R_i}\right)^{2/3}$

Thus, $T_f^{ideal} = T_f^{non\text{-}ideal} \left(2\right)^{2/3}$

$\hookrightarrow \frac{T_f^{non\text{-}ideal}}{T_f^{ideal}} = \left(\frac{1}{2}\right)^{2/3}$

General (Stat. Mech) A. Mueller

V-5

Consider a closed system of N spins in a magnetic field B. Suppose $N/2+s$ of the spins are "up" and $N/2-s$ of the spins are "down" and that the magnetic energy is $U=-2mBs$.

(i) Give the entropy of the system as a function of U.

(ii) Give the temperature of the system as a function of N and U.

You may wish to use $N! \simeq \sqrt{2\pi N}\, e^{N\ln N - N}$ and the fact that the total number of states the configuration above is

$$g(N,s) = \frac{N!}{(N/2+s)!\,(N/2-s)!}$$

Soln:

$$g(N,s) = \sqrt{\frac{2\pi N}{2\pi(N/2+s)\,2\pi(N/2-s)}}\;\frac{e^{N\ln N - N}}{e^{(N/2+s)\ln(N/2+s)+(N/2-s)\ln(N/2-s)-N}}$$

$$g = e^{\sigma} \simeq \text{Prefactor}\; e^{N\ln 2 - 2s^2/N}$$

$$\sigma = N\ln 2 - 2s^2/N = \boxed{N\ln 2 - \frac{U^2}{2m^2B^2N} = \sigma}$$

$$\frac{1}{\tau} = \frac{\partial\sigma}{\partial U} = -\frac{U}{2m^2B^2N}$$

$\tau = kT$

V-6

Aron Pinczuk
12/22/04

General: particle statistics/thermal effects

Consider an ideal gas composed of N He^3 atoms contained in a vessel of volume V. He^3 isotopes have two protons and one neutron in their nuclei. He^3 atoms have spin ½. Describe the kinetic energy states of the He^3 atoms as quantization in a box for a cube of volume $V=L^3$, where L is the length of the cube. Assume, for simplicity, that the system remains gaseous at all temperatures.

(a) Consider the limit of absolute zero of temperature (T=0).

(i) What is the difference between the lowest and highest energy states of single He^3 atoms.

(ii) Obtain an expression for the total kinetic energy as function of N and V.

(iii) What would be the total kinetic energy for a gas composed of He^4 atoms?

(b) Assume that the temperature is raised slightly so that T remains small. The kinetic energy of the He^3 ideal gas is written as $U(T) = U(0) + \Delta U(T)$. Show by qualitative considerations that the leading term in $\Delta U(T)$ is proportional to T^2.

V - 6 (1)

Pinczuk
Thermo/Stat mech.

Part (a)

(i)

The difference is the Fermi energy ε_F
It is calculated by first obtaining the density of states from quantization in a box

$$g(\varepsilon)\,d\varepsilon = \frac{V}{(2\pi)^3}(2s+1)\,dV_k$$

$s = \frac{1}{2}$ is the value of spin and

$$V_k = \frac{4\pi}{3}k^3$$

The answer is $g(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\varepsilon^{1/2}$

The condition

$$\int_0^{\varepsilon_F} g(\varepsilon)\,d\varepsilon = N$$

Yields $\varepsilon_F = \frac{\hbar^2}{2M}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$

where M is the ^3He mass

(ii)

$$U(0) = \int_0^{\varepsilon_F} \varepsilon g(\varepsilon) d\varepsilon = \frac{V}{2\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\int_0^{\varepsilon_F} \varepsilon^{3/2} d\varepsilon$$

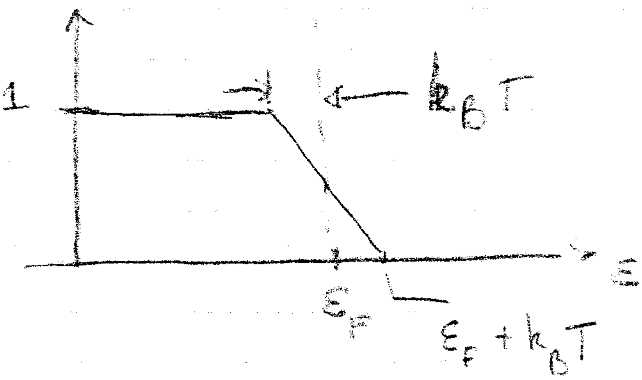
$$= \frac{V}{3\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2} \varepsilon_F^{5/2}$$

(iii)

The $U(0) = 0$ because ^4He are bosons that at $T=0$ drop to the ground state of kinetic energy (equal to zero)

part (b)

Qualitatively the small distortion of the Fermi-Dirac distribution from its $T=0$ value is



The number of ^{3}He atoms that is thermally excited is

$$\Delta N \sim (k_B T)$$

and their energy gain is also proportional to $k_B T$ so that

$$\Delta u(T) \sim \Delta N\,(k_B T) \sim (k_B T)^2$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Thursday, January 13, 2005
11:10 AM – 1:10 PM

General Physics (Part II)
Section 6. Fluids, Optics and General

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 7 included in this section. (You will not earn extra credit by doing additional problems). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 3; Section 6(General Physics) Question 6, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single note sheet on 8 ½ x 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted. Questions should be directed to the proctor.

Good luck!!

Section 6 – Question 1

Choose one colloquium given this semester that particularly interested you and that had direct relevance to a *physics topic*. Briefly describe the topic of this colloquium. Be sure to mention why this topic is important within the context of our current understanding of physics and how the progress described in the colloquium advances that understanding.

Answer in no more than two pages of the blue-book.

Section 6 – Question 2

Consider a cube with refraction index $n > \sqrt{2}$ in the air. Suppose that radiation is emitted isotropically and uniformly at each point inside the cube.

Find the fraction, f, of radiation that will escape the cube.

Section 6 – Question 3

Suppose that a distant planet were to be discovered with an atmosphere consisting solely of hydrogen atoms in their ground states. The atmosphere's number density (n) and scale height (H) are such that a significant fraction (f) of the incident light from the planet's sun is scattered in the atmosphere before reaching the ground.

a) *Very roughly*, what is the fraction, f?
Assume incident solar light has a wavelength, $\lambda_s = 2\pi \times 10^{-5}$ cm, $e^2/\hbar c = 1/137$ and the Bohr radius $\hbar^2/me^2 \approx 5\times10^{-8}$ cm.

b) If the planet's surface were a perfect absorber of all solar light that reaches it, what color planet would we see from a satellite above it? Why?

Section 6 – Question 4

A very large sphere of radius, R is uniformly filled with a very hot, fully ionized hydrogen plasma: N electrons, N protons and $k_B T >> e^4 m/\hbar^2$.

a) *Very roughly*, how long would it take for an energetic photon to get from the center of the sphere to the surface, a distance R away?

b) *Very roughly*, what is the electrical conductivity, σ_c, inside the sphere and how does it depend on the electron number density, N/R^3 ?
Remember: $\vec{j} \equiv \sigma_c \vec{E}$.

Section 6 – Question 5

Consider a one dimensional system in a solid. The lattice constant, *i.e.* the distance between adjacent atoms, is a. Each atom contributes one electron to the conductions band and each electron has spin = ½ .

a) Obtain the Fermi energy and Fermi momentum of the conduction electron system, based on an approximation that each electron is free (the free electron gas model).

b) If we introduce a periodic potential, which has minima at the atomic positions, and maxima halfway between the atoms:

$$U = -U_o \cos\left(2\pi \frac{x}{a}\right)$$

with the atoms at $x = 0, \pm a, \pm 2a, \ldots$, then the parabolic energy dispersion of a free electron is changed. Provide a qualitative drawing of that energy dispersion in the first Brillouin zone, taking into account the effect of the periodic potential. Draw a graph of energy vs. momentum.

c) The maximum energy change (from the parabolic free electron dispersion) would occur for electrons having momentum $k = \pm 2\pi / 2a$ (at the zone boundary). Obtain the amount of the energy change, ΔE, for such an electron in terms of the potential, U_o.

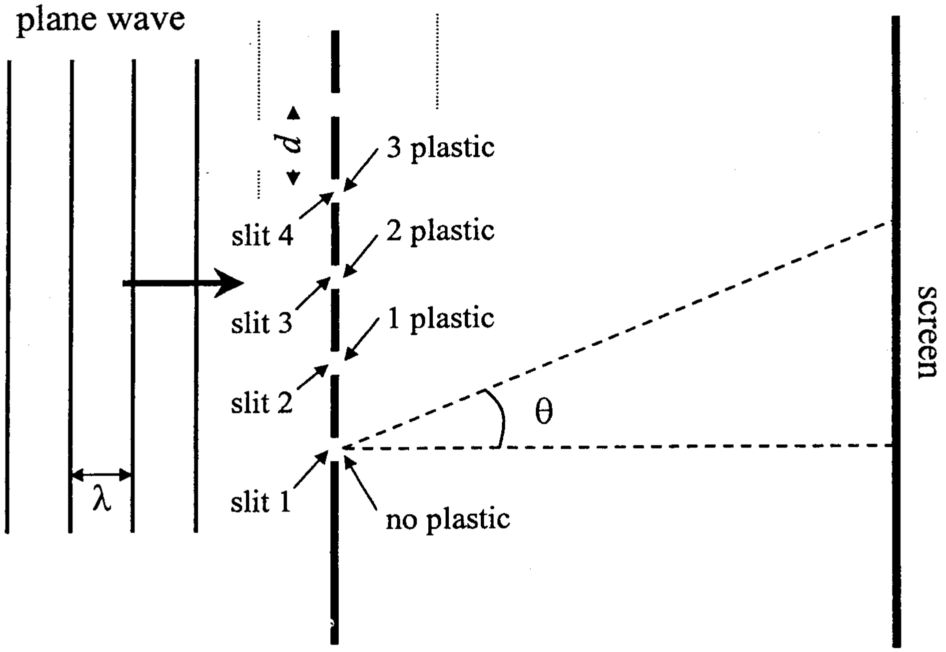
Section 6 – Question 6

The one-dimensional metal system, with a periodic potential, described in the previous problem is unstable against dimerization of adjacent atoms at low temperatures. This phenomenon is called the *Peierls transition*.

a) Provide a brief description of the origin of this instability.

b) There are signatures in lattice and electron systems reflecting Peierls transitions. Describe one experimental measurement for a lattice system and another one for an electron system that detect Peierls transitions. What kind of change in each measurement would you expect between the cases with/without Peierls transitions?

Section 6 – Question 7



Consider the Frauenhofer (far field) diffraction of a monochromatic plane wave (wavelength λ) that is normally incident on a very large number of slits, as shown in the diagram above. Each slit is very long (length L perpendicular to the plane of the diagram) and has the same very narrow width. (Assume that this width is negligibly small). Adjacent slits are separated by the same distance, d.

Consider tiny pieces of transparent plastic which are placed right in front of each slit. Each piece of plastic decreases the intensity of an incident wave by a factor, A ($A < 1$) and shifts the phase of the incident wave by an angle ϕ. The bottom-most slit (slit 1) has no plastic in front of it. The next slit (slit 2) has one piece of plastic in front of it; slit 3 has two pieces of plastic in front of it; and so on. In general, slit N has $(N - 1)$ pieces of plastic in front of it. For this problem, you may assume that there are essentially an infinite number of slits.

The incident wave passes through all of the slits to a distant observing screen. Find the intensity of this diffracted wave at the observing screen as a function of the angle θ shown in the diagram. Plot the intensity as a function of $\sin\theta$ and clearly indicate the angles for which the intensity is a maximum.

2006

Section 6
General Part II
Fluids, optics, etc

VI - 2

Beloborodov 12/8/04

Relativity
Problem 1.

A Martian physicist knows electricity, in particular, Gauss's law and the experimental fact that a charge q at rest is subject to force qE where E is the local electric field. He has not discovered magnetism yet, however, is well educated in relativity. How can he calculate the force acting between two neutral straight long wires A and B bearing equal currents I? The distance betwen the wires d is given and one needs to find the force f per unit length of the wire. [Hint: f doesn't depend on what particles carry the current and their velocity in the wire. The Martian physicist can assume, for instance, that the negative charges are at rest in the lab frame, $v_- = 0$, and the positive charges carry the current with a velocity v_+.]

Solution: The net force acting on, e.g., wire A is the sum of forces acting on its + and − charges $E = 0$ in the lab frame (the wires are neutral), so the force acting on the non-moving (−) charges is zero. The challenge is to calculate the force acting on the moving (+) charges. The Martian physicist can do that in the rest frame F of the charges and needs to find the electric field E created by wire B in frame F. The field is created because wire B has a line charge density $\lambda \neq 0$ in frame F:

$\lambda = \lambda_+ - \lambda_-$, where $\lambda_\pm$ are the line charge densities of the $\pm$ charges. They are related to the corresponding densities in the lab frame $\lambda_{\pm,\mathrm{lab}}$ by

$$\lambda_- = \gamma\lambda_{-,\mathrm{lab}}, \quad \lambda_+ = \frac{\lambda_{+,\mathrm{lab}}}{\gamma}, \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v_+}{c}.$$

Using the wire neutrality in the lab frame, $\lambda_{+,\mathrm{lab}} = \lambda_{-,\mathrm{lab}}$, one finds

$$\lambda = \left(\frac{1}{\gamma} - \gamma\right)\lambda_{+,\mathrm{lab}} = -\frac{(\gamma^2-1)}{\gamma}\frac{I}{c\beta} = -\gamma\beta\frac{I}{c}.$$

From Gauss's law, one finds E in wire A (created by wire B),

$$E = \frac{2\lambda}{d} = -2\gamma\beta\frac{I}{cd}.$$

The negative sign shows that wire B will attract the + charges of wire A. The attraction force is

$$f = \lambda_+ E = -\frac{I}{\gamma\beta c}E = -\frac{2I^2}{c^2 d}.$$

This force is measured in frame F. It remains the same after transformation to the lab frame, which can be seen from the fact that γ did not enter the expression for f. [Formal proof: Lorentz transformation of 4-force $dp^\alpha/ds = (0,0,f,0)$ to the lab frame gives $dp^\alpha_{\mathrm{lab}}/ds = (0,0,f,0)$, and hence $f_{\mathrm{lab}} = f$.]

General **Problem 2.**

Consider a cube with refraction index $n > \sqrt{2}$ in the air. Suppose radiation is emitted isotropically and uniformly at each point inside the cube. Find the fraction f of radiation that will escape the cube.

Solution: The cube has six faces with normals $\pm\mathbf{e}_x$, $\pm\mathbf{e}_y$, and $\pm\mathbf{e}_z$. A light ray propagating in a direction $\mathbf{k}$ will escape through faces $\pm\mathbf{e}_x$ if $|\mathbf{k}\cdot\mathbf{e}_x| > \sqrt{1-n^{-2}}$, through faces $\pm\mathbf{e}_y$ if $|\mathbf{k}\cdot\mathbf{e}_y| > \sqrt{1-n^{-2}}$, and through faces $\pm\mathbf{e}_z$ if $|\mathbf{k}\cdot\mathbf{e}_z| > \sqrt{1-n^{-2}}$. On the unit sphere $|\mathbf{k}| = 1$ there are six regions defined by these three conditions, which do not intersect (since $n > \sqrt{2}$) and occupy the total solid angle

$$\Omega_{\mathrm{esc}} = 6 \times 2\pi\left(1 - \sqrt{1-n^{-2}}\right).$$

The escaping fraction of radiation is given by

$$f = \frac{\Omega_{\mathrm{esc}}}{4\pi} = 3\left(1 - \sqrt{1-n^{-2}}\right).$$

Mel Ruderman

① General : (transmission and diffusion of light)

VI - 3

Suppose a distant planet were to be discovered whose atmosphere consisted of hydrogen atoms in their ground states. The atmosphere's number density (n) and scale height (H) are such that a significant fraction (f) of incident light from the sun is scattered in that atmosphere before reaching the ground

a) Very roughly what is that fraction?

(8/10)

Assume incident solar light has a wavelength /2 $\equiv \lambda_s = 10^{-5}$ cm, $e^2/\hbar c = 1/137$, and the Bohr radius $\hbar^2/me^2 \approx 5 \times 10^{-8}$ cm.

(2/10)

b) If that planet's surface were a perfect absorber of all solar light which reaches it what color planet would we see (from a satellite above it)? Why?

Ruderman
General question #1
transmission + diffusion of light
solution p. 1

(1) Answer

a) $f \sim n_H \sigma$ $\sigma \equiv$ crosssection for scattering of solar light by an H-atom

Approximate H atom as an harmonic oscillator with ~~of~~ frequency ω_0. Then for frequency ω light

$$\sigma \sim \sigma_T \left(\frac{\omega^2}{\omega_0^2 - \omega^2}\right)^2 \quad \text{with}$$

σ_T the Thomson cross section $\sim \left(\frac{e^2}{mc^2}\right)^2$

[There are many ways to get this

e.g. $m\ddot{\vec{x}} = e\vec{E} - m\omega_0^2 \vec{x}$

power radiated out by perturbation (by $e\vec{E}$) from EM wave

$$\ddot{x} = \frac{\omega^2 eE/m}{\omega^2 - \omega_0^2}$$

$\sigma \propto$ radiated power $\propto (\ddot{x})^2$

so $\sigma = A \left(\frac{\omega^2}{\omega^2 - \omega_0^2}\right)^2$ But $\sigma = \sigma_T$ if $\omega_0 = 0$.]

We are in Rayleigh limit $\omega \ll \omega_0$

$$\therefore \boxed{f \sim n_H \sigma_T \left(\frac{\lambda\!\!\!^{-}_s}{\lambda\!\!\!^{-}_0}\right)^4} \qquad \left[\lambda\!\!\!^{-}_0 \equiv c/\omega_0\right]$$

$\omega_0 = \frac{e^2}{\hbar c}\frac{c}{a_B}$ = angular freq of e in lowest Bohr orbit

$\left(= \frac{v}{a_B} = \left(\frac{e^2}{\hbar c}\right)(c) / a_B\right)$

a_B $me^4/\hbar^3$

or use main absorption line

$$\hbar\omega_0 \sim \frac{me^4}{\hbar^2}$$

Rydberg

Ruderman

General Question # 1
transmission + diffusion of light
Solution - p. 2

So final result is

$$f = n H \sigma_T \left(\frac{\lambda_s}{a_B} \cdot \frac{e^2}{\hbar c} \right)^4$$

with

$$\sigma_T \sim \left(\frac{e^2}{mc^2} \right)^2$$

b) Blue light scatters more than red light in our Rayleigh regime because $\lambda_{Blue} < \lambda_{red} << \lambda_0$ so

planet would look blue.

M. Ruderman

(2)

VI - 4

General i (diffusion and conductivity)

A very large sphere of radius R is uniformly filled with a very hot ~~hydrog~~. fully ionized, hydrogen plasma. (N electrons, N protons, and $k_B T \gg e^4 m/\hbar^2$)

(4/10) a) Very roughly, how long would it Take for an energetic photon To get from the center of the sphere To the surface R away?

(4/10) b) Very roughly what is the electrical conductivity (σ_c) inside this sphere? ($\vec{J} \equiv \sigma_c \vec{E}$). How does it depend on the electron number density (N/R^3)?

(2/10) c) Electrical currents within the plasma are the source of a weak magnetic field (B) which fills the interior. Very roughly how long would it Take for B To die out because of ohmic (Eddy current) dissipation? (Assume B initially varies on a scale R.)

Answer:

a) $t \sim \frac{R^3}{c\Lambda}$ ← R^2 I presume...

Λ = mean free path $\sim \frac{1}{n_e \sigma_T} \sim \frac{R^3}{N\sigma_T}$

($n_e \sim \frac{N}{R^3}$; σ_T = Thomson crosssection)

$$\therefore \boxed{t \sim \frac{\sigma_T N}{Rc}}$$

---

b) $\sigma_c \sim e\left(\frac{N}{R^3}\right) \times \left(\frac{eE}{m_e}\right) \times \tau_c$

($\frac{N}{R^3}$ = n_e; $\frac{eE}{m_e}$ = acc.; τ_c = time between collisions)

A good student (especially with hint in question) will note that σ_c can't depend upon the electron number density N/R^3 because $\tau_c \propto \frac{1}{\text{proton number density}} \sim \frac{1}{N/R^3}$.

then Dimensional Analysis alone gives

$$\boxed{\sigma_c \sim \left(\frac{kT}{m_e}\right)^{1/2}\left(\frac{kT}{e^2}\right)}$$

(m_e = electron mass)

or he/she could work it out from

$$\overline{\vec{v}_e} = \frac{e\vec{E}}{m}\tau_c \sim \frac{e\vec{E}}{m}\left(\frac{R^3}{N}\right)\left(\frac{1}{v_T \sigma_c}\right)$$

v_T = Thermal velocity $\sim \left(\frac{kT}{m_e}\right)^{1/2}$

σ_c = coulomb scattering cross section $\left(\frac{e^2}{\varepsilon}\right)^2 \sim \frac{e^4}{(kT)^2}$

Then $\sigma_c \sim \frac{eN}{R^3} \times \overline{v_e}$ gives the $\boxed{}$ above

VI - 5
Solution

January 12, 2005

Solutions:

Section 6, Question 5, 1-d metal, Tomo Uemura.

a)

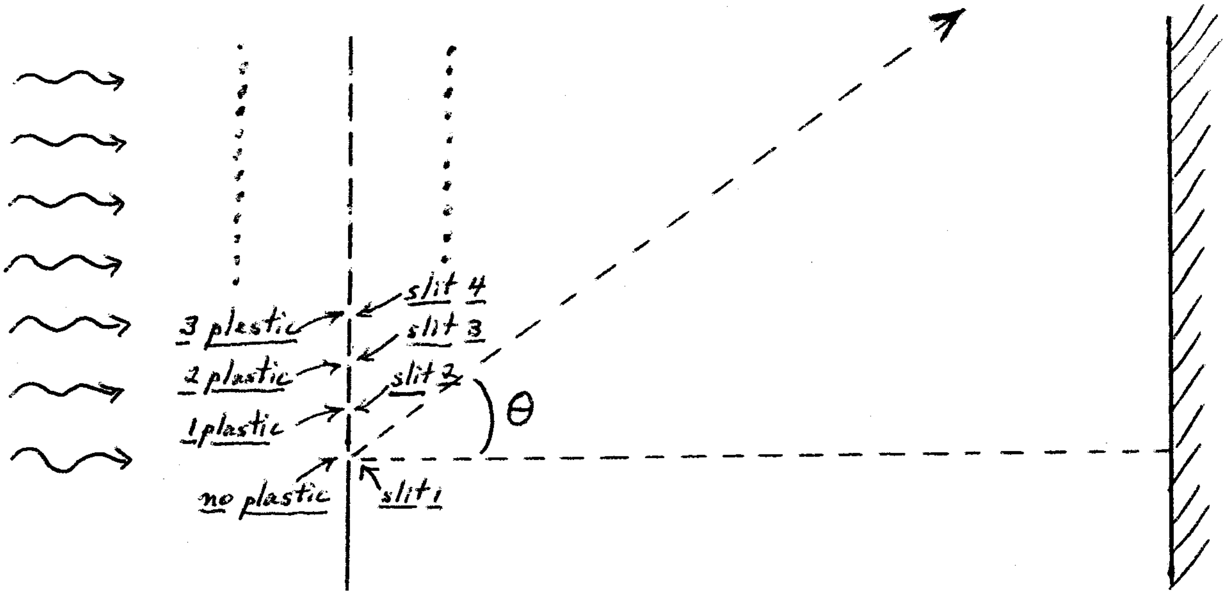
periodic boundary condition. lattice constant a. length L. $\Delta k = 2\pi/L$. Suppose we fill up to the Fermi momentum $\pm k_F$. Number of states within the Fermi momentum, with spin up/down digeneracy: $N = 2(spin) \times 2k_F/\Delta k = 4k_F L/2\pi$. This equals to the number of conduction electron $N = L/a$. Thus $k_F = (1/4) \times (2\pi/a)$; i.e., a half way to the zone boundary $2\pi/2a$. Fermi energy $\epsilon_F = \hbar^2 k_F^2/2m$.

b) The periodic potential with the periodicity of the lattice constant would modify the parabollic dispersion, and create the energy gap at the zone boundary $k_{ZB} = 2\pi/2a$. See Figure 1.

c) This potential would act to the wave functions at the zone boundary $exp(ik_{ZB}x)$ and $exp(-ik_{ZB}x)$, mix them, leading to $\phi_{cos} = cos(k_{ZB}x)$ which has the maximum amplitude at the atomic position $x = 0$ and $\phi_{sin} = sin(k_{ZB}x)$ which has the minimum amplitude at the atomic position. The charge density would be ϕ^2, which is a sinusoidal function having periodicity of $2k_{ZB} = 2\pi/a$. The periodic potential has the same periodicity as the charge density, with the full amplitude U_o. Multiplying the charge density and the periodic potential, integrating over the Brillouin zone, would give the energy saving of $U_o/2$ to the cosine wave, and energy increase of the same amount to the sine wave. Thus $\Delta E = \pm U_o/2$ at the zone boundary. See Figure 2.

VI - 7
No solution provided

Ph.D. Qualifying Examination
Problem on Optics for the General Physics Part
Allan Blaer



Consider the Frauenhofer (far field) diffraction of a monochromatic plane wave (wavelength λ) that is normally-incident on a very large number of slits, as shown in the diagram. Each slit is very long (length L perpendicular to the plane of the diagram) and has the same very narrow width. (Assume that the slits' widths are negligibly small.) Adjacent slits are separated by the same fixed distance d. Consider tiny pieces of plastic which are placed right in front of each slit. Each piece of plastic decreases the intensity of an incident wave by the factor A ($A < 1$) and shifts the phase of an incident wave by phase angle ϕ. The bottom-most slit (slit number 1) has no plastic in front of it. The next slit (slit number 2) has 1 piece of plastic in front of it. The next slit (slit number 3) has 2 pieces of plastic in front of it. In general, slit number N has (N-1) pieces of plastic in front of it. For this problem, you may assume that there are essentially an infinite number of slits present. The incident wave passes through all of the slits, and the resulting diffracted wave then travels from the slits to a distant observing screen. Find the intensity of this diffracted wave at the observing screen as a function of the angle θ shown in the diagram. Plot the intensity as a function of $\sin\theta$, and clearly indicate the angles for which the intensity is a maximum.

Rudermann
General ~~Question # 2~~ diffusion + conductivity
Solution p. 2

VI - 4

c) $$t \sim \frac{R^2 \sigma_c}{c^2}$$

with the σ_c of b)

Tomo Uemura   general II

Problem I.

Solid State Physics:
Re: 1-dimensional metal, band-gap for periodic potential, and Peierls transition

Problem:

We consider a one dimensional system of solid. The lattice constant, i.e., the distance of adjacent atom, is a.
Each atom contributes one electron to conduction band.
Each electron has spin S = 1/2.

(1) Obtain Fermi energy and Fermi momentum of the conduction electron system, based on an approximation that each electon is a free electron (free electron gas model).

(2) If we introduce a periodic potential, which has a minimum at the atomic position, and maximum in the center of the adjacent atoms, such as
$$U = - \cos(2\pi x/a)$$ (for the position of atom at x = 0, a, 2a, -a, -2a, ...)
then the parabolic energy dispersion of a free electron is changed. Provide a qualitative drawing of the energy dispersion of electrons in the first Brillouin zone taking into account the effect of this periodic potential. Draw a graph of energy versus momentum.

(3) The maximum energy change (from the parabolic free electron dispersion) would occur for electrons having momentum of $k = +/- 2\pi / 2a$ (at the zone boundary). Obtain the amount of the energy change for such an electron. Give an answer in a way such as ΔE = U/5 or 10 U or ....

(4) Such a one-dimensional metal system is unstable against dimerization of adjacent atoms at low temperatures. This phenomenon is called Peierls transition.
Provide a brief desctription of the origin of this instablility.

(5) There are signatures in lattice system and electron system reflecting Peierls transition. Describe one experimental measurement for lattice system, and another one for the electron system, which detect Peierls transition. What kind of change in each measurement would you expect between the cases with/without Peierls transition ?