Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 10, 2005
9:00 AM – 11:00 AM

Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 1; Section 1(Classical Mechanics) Question 3, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

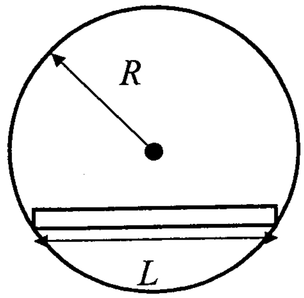
Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

Section 1 – Question 1

A stick of uniform density, mass, M, and length, L is constrained to move such that its ends rest on the inside of a fixed vertical, circular ring of radius R, as shown below.

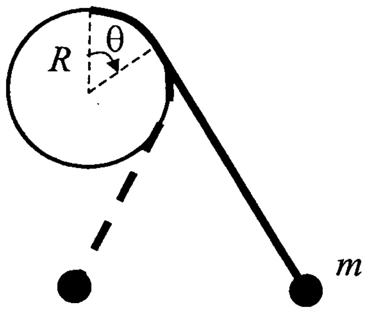


Find the frequency for *small* oscillations about the stick's equilibrium position.

You can neglect friction between the stick and the ring.

Section 1 – Question 2

Consider a pendulum built from a mass m attached to one end of a massless, extensionless string, whose other end is attached to the uppermost point of a fixed, vertical disk of radius R, as shown in the figure below. Assume that the total length of the string is l and that $\pi R < l$.



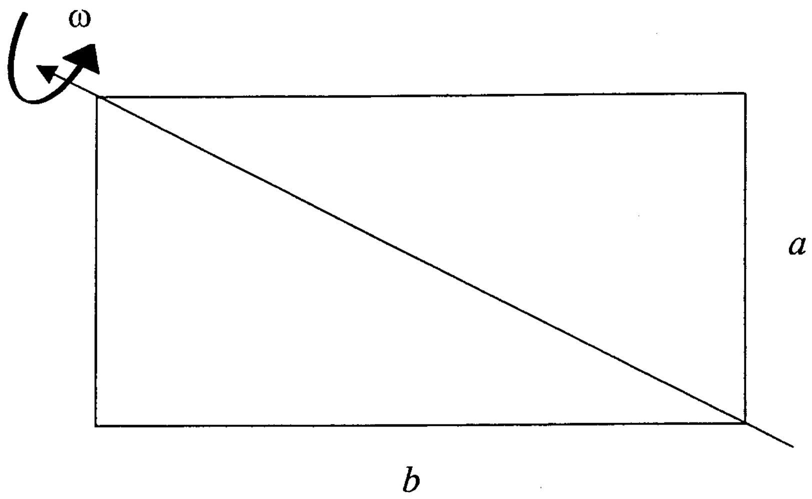
a) Find the equations of motion in terms of the angle, θ, as shown in the figure.

b) What is the equilibrium angle, θ_0?

c) Find the frequency of oscillations about the equilibrium angle.

Section 1 – Question 3

A uniform rectangular object of mass m with sides a and b ($b > a$) and negligible thickness rotates with constant angular velocity ω about a diagonal through the center. (For this problem, ignore gravity.)



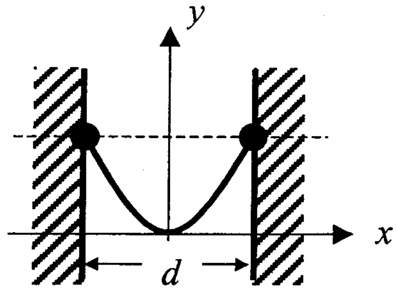
a) What are the principle axes and moments of inertia?

b) What is the angular momentum vector in body coordinate system?

c) What external torque must be applied to keep the object rotating with constant angular velocity around the diagonal?

Section 1 – Question 4

A chain with uniform mass density ρ (per unit length) hangs between two points on two walls as shown below. Assume that these two points are level.



(a) Find the shape of the chain. Apart from an arbitrary additive constant, the function describing the shape should also contain an unknown constant.

(b) Find an equation for this unknown constant in terms of the length l of the chain and the separation d between the walls.

Section 1 – Question 5

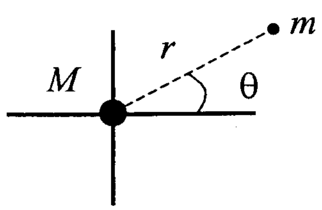
The gravitational potential near a black hole of mass, M and Schwarzschild radius, a can be described by a modified classical potential:

$$U(r) = -\frac{GM}{(r-a)}$$

a) Find an expression for the force acting on a particle of mass, m in this gravitational potential.

b) Expand your answer from part a) to find the lowest order correction to the classical gravitational force when the test particle is at a distance from the black hole large enough to satisfy the condition, $a / r << 1$.

c) Develop a solution for the orbital motion of a test particle near the black hole using the force from part b) – just the classical and first order correction terms. Express your answer in the form $r = r(\theta)$, where θ is the angle between a fixed axis and the radius vector to the particle, as shown below.
You need only find the solutions periodic in θ.



J. Parsons I-1

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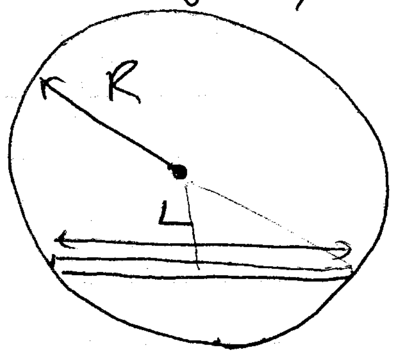
Oscillation problem

A stick of uniform density, mass M, length L is constrained to move such that its ends rest on the inside of a circular ring of radius R. Find the frequency for small oscillations about its equil^m position.



(You can neglect friction between the stick and ring.)

Answer

about CM $\rightarrow$ $I_{cm} = \frac{M}{L}\int_{-L/2}^{+L/2} x^2\,dx = \frac{M}{L}\left.\frac{x^3}{3}\right|_{-L/2}^{+L/2} = \frac{1}{12}ML^2$

For rotation about centre of ring

$$I = I_{cm} + Ml^2 \quad \text{with } l^2 = R^2 - \left(\frac{L}{2}\right)^2$$
$$= \frac{1}{12}ML^2 + M\left(R^2 - \frac{L^2}{4}\right) = M\left(R^2 - \frac{L^2}{6}\right)$$

The $\tau = I\ddot{\theta} = Mgl\sin\theta \simeq Mgl\theta$ for small θ

$\hookrightarrow Mgl\theta \simeq -M\left(R^2 - \frac{L^2}{6}\right)\ddot{\theta}$

$\hookrightarrow$ SHM with $\omega = \sqrt{\frac{gl}{(R^2 - L^2/6)}} = \sqrt{\frac{g\sqrt{R^2 - \frac{L^2}{4}}}{R^2 - L^2/6}}$

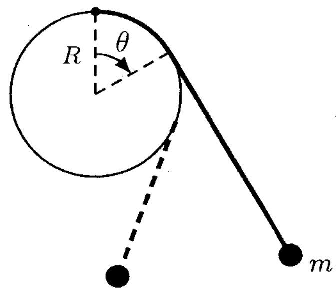
I - 2

DEC 2004

Qualifying exam
Lagrangian mechanics and relativity
Eduardo Pontón

mechanics

1. Consider a pendulum built from a mass m attached to one end of a massless, extensionless string, whose other end is attached to the uppermost point of a vertical disk of radius R, as shown in the figure. Assume that the total length of the string is l and that $\pi R < l$.



   (a) Find the equations of motion in terms of the angle θ as shown in the figure.

   (b) What is the equilibrium angle θ_0? Find the frequency of small oscillations about this position.

Relativity

2. Meson factories produce secondary π^+-meson beams from collisions of high energy protons with nuclear targets. The π^+-meson decays in flight mainly through the channel $\pi^+ \to \mu^+ + \nu_\mu$.

   (a) Without approximation derive formulas for the muon and neutrino energies, E_μ and E_ν, in the pion's rest frame, in terms of the pion and muon masses (neglect the neutrino mass).

   (b) The π^+ has spin zero. What is the angular distribution of the μ^+ in the pion's rest frame?

   (c) What is the range of possible energies of the muon, E'_μ, in the laboratory frame? To what physical situations do the maximum and minimum values correspond?

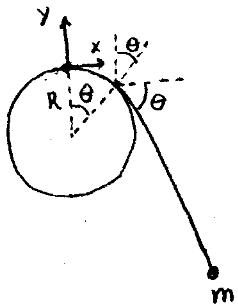
   (d) Making use of your results in parts (b) and (c) obtain the probability distribution of the laboratory μ^+ energy.

Lagrangian mechanics and relativity

Eduardo Pontón

Lagrangian mechanics

1) a)



Since the string extends tangentially to the disk, the geometry is as shown in the figure. Also, the position of m is

$$x = R\sin\theta + (\ell - R\theta)\cos\theta$$

$$y = -R(1-\cos\theta) - (\ell - R\theta)\sin\theta$$

where $(\ell - R\theta)$ is the length of the string that extends from the disk

Therefore,

$$\dot{x} = \cancel{R\dot{\theta}\cos\theta} + \cancel{(-R\dot{\theta})\cos\theta} - (\ell - R\theta)\dot{\theta}\sin\theta = -(\ell - R\theta)\dot{\theta}\sin\theta$$

$$\dot{y} = \cancel{-R\dot{\theta}\sin\theta} - \cancel{(-R\dot{\theta})\sin\theta} - (\ell - R\theta)\dot{\theta}\cos\theta = -(\ell - R\theta)\dot{\theta}\cos\theta$$

$$\Rightarrow \quad L = \frac{1}{2}m(\ell - R\theta)^2\dot{\theta}^2 - mg\left[-R(1-\cos\theta) - (\ell - R\theta)\sin\theta\right]$$

$$= \frac{1}{2}m(\ell - R\theta)^2\dot{\theta}^2 + mgR(1-\cos\theta) + mg(\ell - R\theta)\sin\theta$$

$$\Rightarrow \quad \frac{\partial L}{\partial \dot{\theta}} = m(\ell - R\theta)^2\dot{\theta} \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -2m(\ell - R\theta)R\dot{\theta}^2 + m(\ell - R\theta)^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = m(\ell - R\theta)(-R)\dot{\theta}^2 + \cancel{mgR\sin\theta} + \cancel{mg(-R)\sin\theta}$$

$$+ mg(\ell - R\theta)\cos\theta$$

$$= -m(\ell - R\theta)\left[R\dot{\theta}^2 - g\cos\theta\right]$$

Lagrange equation: [cancel out a factor $m(\ell - R\theta)$]

$$(\ell - R\theta)\ddot{\theta} - 2R\dot{\theta}^2 + [R\dot{\theta}^2 - g\cos\theta] = 0$$

$$\Rightarrow \boxed{(\ell - R\theta)\ddot{\theta} - R\dot{\theta}^2 - g\cos\theta = 0}.$$

b) In equilibrium: $\dot{\theta} = \ddot{\theta} = 0 \Rightarrow \cos\theta = 0 \Rightarrow \boxed{\theta_0 = \frac{\pi}{2}}$ (makes sens

Assume $\theta = \frac{\pi}{2} + \varepsilon$, $\varepsilon \ll 1$

$$\Rightarrow \dot{\theta} = \dot{\varepsilon}, \quad \ddot{\theta} = \ddot{\varepsilon}, \quad \cos\theta = -\sin\varepsilon \approx -\varepsilon$$

$$\Rightarrow (\ell - R\tfrac{\pi}{2})\ddot{\varepsilon} + g\varepsilon = 0$$

neglect ε: higher order; also $\dot{\theta}^2 = \dot{\varepsilon}^2$ is higher '

So the frequency of small oscillations is

$$\boxed{\omega = \sqrt{\frac{g}{\ell - R\frac{\pi}{2}}}}$$

I- 3

NOV 8 ... 04

Mechanics Problem

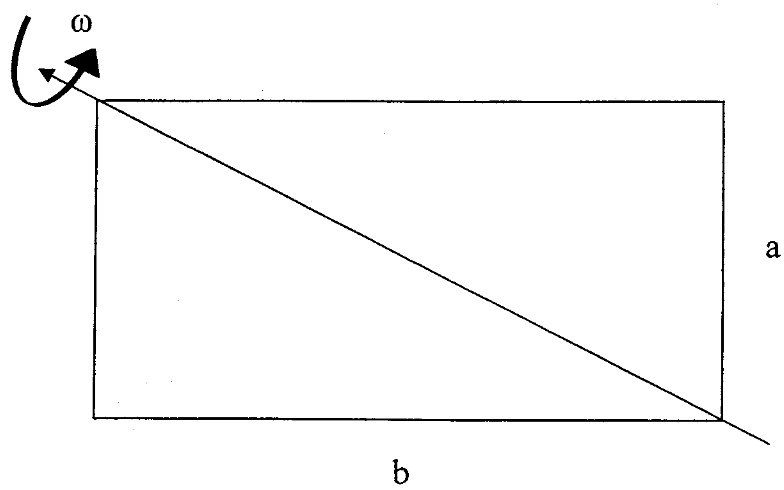
M. Shaevitz

A uniform rectangular object of mass m with sides a and b (b > a) and negligible thickness rotates with constant angular velocity ω about a diagonal through the center. (For this problem, ignore gravity.)

a) What are the principle axes and moments of inertia?

b) What is the angular momentum vector in body coordinate system?

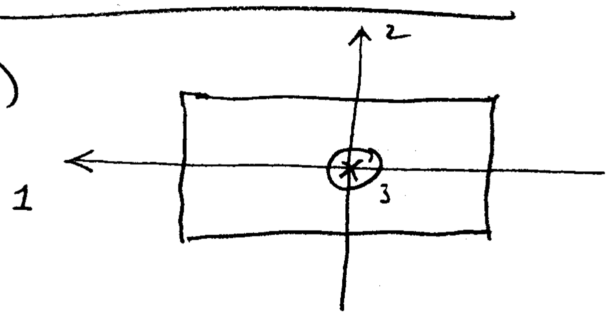
c) What external torque must be applied to keep the object rotating with constant angular velocity around the diagonal?



Mechanics Problem

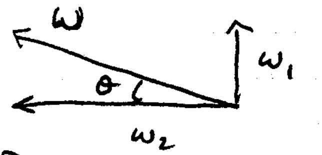
M. Shaevitz
I-3 solution

a)



$$I_1 = \frac{ma^2}{12} \qquad I_2 = \frac{mb^2}{12} \qquad I_3 = I_1 + I_2 = \frac{m(a^2+b^2)}{12}$$

b) $$\omega_1 = \frac{\omega b}{(a^2+b^2)^{1/2}} \qquad \omega_2 = \frac{\omega a}{(a^2+b^2)^{1/2}}$$



$$\omega_3 = 0 \qquad \vec{\omega} = \frac{\omega}{(a^2+b^2)^{1/2}} (b, a, 0)$$

$$\vec{L} = I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3$$

$$= \left(\frac{ma^2}{12}\right)\frac{\omega b}{(a^2+b^2)^{1/2}} \hat{e}_1 + \left(\frac{mb^2}{12}\right)\left(\frac{\omega a}{(a^2+b^2)^{1/2}}\right) \hat{e}_2 + 0\, \hat{e}_3$$

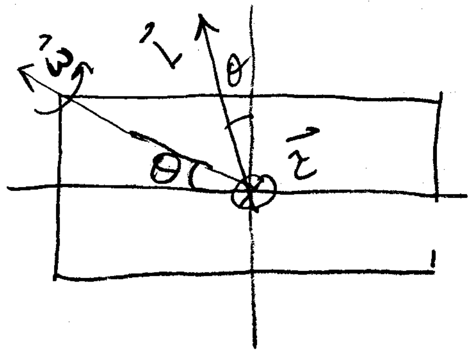
$$\boxed{\vec{L} = \frac{mab\,\omega}{12(a^2+b^2)^{1/2}} (a, b, 0)}$$

c) In body coordinate system $\vec{\omega}$ = constant

$$\vec{\tau} = \cancelto{0}{\frac{d\vec{L}}{dt}} + \vec{\omega} \times \vec{L} = \vec{\omega} \times \vec{L}$$

$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & 0 \\ L_1 & L_2 & 0 \end{vmatrix} = (\omega_1 L_2 - \omega_2 L_1)\, \hat{e}_3$$

$$\boxed{\vec{\tau} = \frac{m\omega^2 ab}{12(a^2+b^2)} (b^2 - a^2)\, \hat{e}_3}$$



Statics

Aprile - I - 4

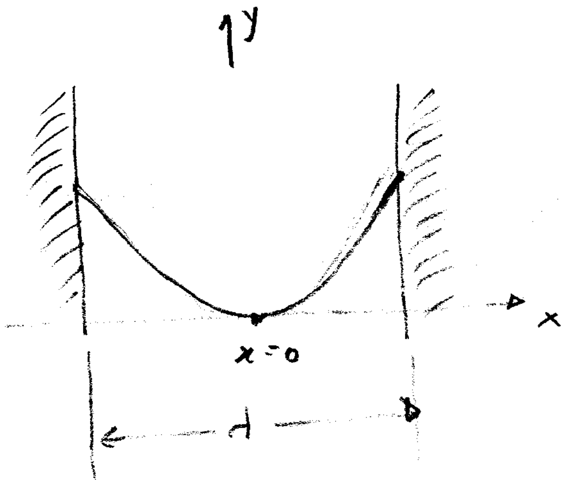
DEC 2004

a chain with uniform mass density ρ / unit length hangs between two given points on two walls. Assume that these two points are are level.

(a) Find the shape of the chain.

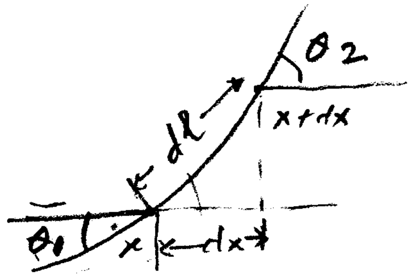
Apart ~~aside~~ from an arbitrary additive constant, the function describing the shape should also contain an unknown constant.

(b) Find an equation for this unknown constant in terms of the length l of the chain and the separation d between the walls.



Solution:

ⓐ



Consider the equilibrium of a section of the chain between x and $x+dx$.

The length of this section is $d\ell = \dfrac{dx}{\cos\theta_1}$

& the weight is $dw = \rho\, d\ell\, g = \rho g \dfrac{dx}{\cos\theta_1}$.

Let $T(x)$ be the tension at x & $T(x+dx)$ that at $x+dx$.

Then: Balance of forces
- Horizontal: $T(x+dx)\cos\theta_2 = T(x)\cos\theta_1$.
- Vertical: $T(x+dx)\sin\theta_2 = T(x)\sin\theta_1 + \dfrac{\rho g}{\cos\theta_1}dx$.

Squaring & adding,

$$T^2(x+dx) = T^2(x) + 2\rho g\, T(x)\tan\theta_1 + O(dx^2)$$

Recall: $T(x+dx) = T(x) + T'(x)dx + O(dx^2)$

$$\Rightarrow T^2(x) + 2T(x)T'(x)\,dx = T^2(x) + 2\rho g\, T(x)\tan\theta_1 \quad [\text{Dropping } O(dx^2)]$$

$$\Rightarrow T'(x) = \rho g \tan\theta_1 .$$

But $\tan\theta_1 = \dfrac{dy}{dx} \Rightarrow T'(x) = \rho g\, y'(x)$

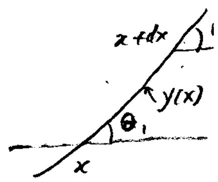
$$\Rightarrow \boxed{T(x) = \rho g\, y(x) + C_1} \quad \ldots\ldots ①$$

We still have to eliminate T in terms of purely geometric quantities.

③ I-4

$$\left.\begin{aligned} \text{again, } y'(x) &= \tan\theta_1 \\ \text{\& } y'(x+dx) &= \tan\theta_2 \end{aligned}\right\} \begin{aligned} &\Rightarrow \cos\theta_1 = \frac{1}{\sqrt{1+\tan^2\theta_1}} = \frac{1}{\sqrt{1+(y'(x))^2}} \\ &\Rightarrow \cos\theta_2 = \frac{1}{\sqrt{1+\tan^2\theta_2}} = \frac{1}{\sqrt{1+(y'(x+dx))^2}} \end{aligned}$$

So, the condition for horizontal equilibrium is:



$$T(x+dx)\,\frac{1}{\sqrt{1+\underbrace{(y'(x+dx))^2}_{\approx\, y'^2(x)+2y'(x)y''(x)\,dx}}} = T(x)\,\frac{1}{\sqrt{1+(y'(x))^2}}$$

$$\Rightarrow \left(T(x)+T'(x)\,dx\right)\frac{1}{\sqrt{1+y'^2(x)}}\left(1+\frac{2y'y''dx}{1+y'^2}\right)^{-1/2}$$

$$= T(x)\,\frac{1}{\sqrt{1+y'^2(x)}}$$

$$\Rightarrow (T+T'dx)\,\frac{1}{\sqrt{1+y'^2}}\left(1-\frac{y'y''dx}{1+y'^2}\right) = T\,\frac{1}{\sqrt{1+y'^2}}$$

$$\Rightarrow \frac{T'}{\sqrt{1+y'^2}} - \frac{y'y''T}{(1+y'^2)^{3/2}} = 0$$

$$\Rightarrow \frac{T'}{T} = \frac{y'y''}{1+y'^2} = \frac{1}{2}\,\frac{d(1+y'^2)}{(1+y'^2)}.$$

$$\Rightarrow 2\ln T = \ln T^2 = \ln(1+y'^2) + \text{const}.$$

$$\Rightarrow \boxed{T^2 = c_2^2\,(1+y'^2)} \quad \text{-----} \quad ②.$$

constant of integration.

Aprile Statics Solution - 3

Eliminate T from ① & ②:

$$(\rho g y + c_1)^2 = c_2^2(1+y'^2).$$

$$\Rightarrow 1+y'^2 = \left(\frac{\rho g}{c_2} y + \frac{c_1}{c_2}\right)^2 = \underbrace{\left(\frac{\rho g}{c_2}\right)^2}_{\equiv\, \alpha^2} \Big(y + \underbrace{\frac{c_1}{\rho g}}_{\equiv\, \beta}\Big)^2$$

$$\Rightarrow \boxed{1+y'^2 = \alpha^2(y+\beta)^2} \quad \ldots ③$$

We show below that $y(x)+\beta = \frac{1}{\alpha}\cosh\alpha(x+\gamma)$, where α, β, γ are constants, is a solution of ③

Now $1+\sinh^2\theta = \cosh^2\theta$

So, let $\boxed{y+\beta \equiv \frac{1}{\alpha}\cosh\alpha(x+\underbrace{\gamma}_{\text{constant}})} \quad \ldots ④$

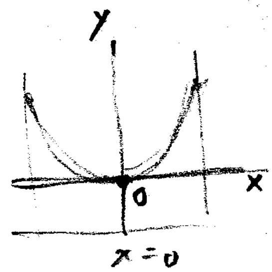
This implies $\Rightarrow y' = \sinh\alpha(x+\gamma)$

$$\Rightarrow 1+y'^2 = \cosh^2\alpha(x+\gamma) = \alpha^2(y+\beta)^2.$$

So $\boxed{y(x)+\beta = \frac{1}{\alpha}\cosh\alpha(x+\gamma) \text{ is indeed a solution of ③}}$ ~~of ③~~

Evaluating the constants:

Now if $y=0$ at $x=0$, (lowest point of chain).



$$0+\beta = \frac{1}{\alpha}\cosh\alpha\gamma.$$

But $\alpha\beta = \frac{\rho g}{c_2}\cdot\frac{c_1}{\rho g} = \frac{c_1}{c_2}$.

$$\Rightarrow \boxed{\cosh\frac{\rho g \gamma}{c_2} = \frac{c_1}{c_2}.} \Rightarrow c_1 = c_2$$

also, at $x=0$, $y'(0)=0$ (tangent is horizontal)

④ $\Rightarrow y'(x) = \sinh\alpha(x+\gamma) \Rightarrow 0 = \sinh\alpha\gamma \Rightarrow \boxed{\gamma = 0} \Rightarrow \boxed{c_1 = c_2}.$

as $\alpha \neq 0$ $\Downarrow$ $\cosh\frac{\rho g\gamma}{c_2}{}_{=0} \Rightarrow \boxed{\alpha\beta = \frac{c_1}{c_2} = 1}$

$$\Rightarrow \boxed{\beta = \frac{1}{\alpha}}$$

So, $y(x) + \frac{1}{\alpha} = \frac{1}{\alpha}\cosh\alpha x$.

$$\Rightarrow \boxed{y(x) = \frac{1}{\alpha}(\cosh\alpha x - 1)} \quad \ldots (5)$$

Shape of the chain (α is the undetermined constant).

Equation for α:

(6) Length of the chain

$$\ell = \int_{-d/2}^{d/2} d\ell = \int_{-d/2}^{d/2} \frac{dx}{\cos\theta_1} = \int_{-d/2}^{d/2} dx \sqrt{1+y'^2}$$

$$= 2\int_0^{d/2} dx \sqrt{1+y'^2}$$

But $y' = \sinh\alpha x$

$$\Rightarrow 1+y'^2 = \cosh^2\alpha x$$

$$\Rightarrow \quad \ell = 2\int_0^{d/2} dx \cosh\alpha x = \frac{2}{\alpha}\int_0^{d/2} dx \frac{d}{dx}\sinh\alpha x$$

$$\ell = \frac{2}{\alpha}\left[\sinh\frac{\alpha d}{2}\right]$$

$$\Rightarrow \boxed{\sinh\frac{\alpha d}{2} = \frac{\alpha \ell}{2}} \quad \ldots (6)$$

This gives α in terms of ℓ & d.

Hailey = gravity/Central force

Hailey I,75
mechanics - Question

The gravitational potential near a black hole of mass M can be described by a modified classical potential $U(r) = -\frac{GM}{(r-a)}$

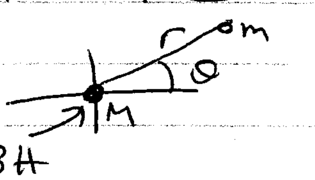
where M is the mass of the black hole and a its Schwarzschild radius.

a.) Find an expression for the force acting on a particle of mass m in this gravitational potential.

b.) Expand your answer from part a to find the lowest order correction to the classical gravitational force - when the test particle is at a distance from the black hole large enough to satisfy the condition $a/r << 1$.

c.) Develop a solution for the orbital motion of a test particle near the black hole using the force from part b - just the classical and first order correction terms. Express your answer in the form $r = r(\theta)$ where θ is an angle between a fixed axis and the radius vector to the particle.

You need only find the solutions periodic in θ.



Sol'n : gravity Central Force

Hailey
Mechanics - I-V
solution - p. 1

a.) $U = -\frac{GM}{r-a}$

$f(r) = -\frac{dU}{dr} m = -\frac{GMm}{(r-a)^2}$

b.) $f(r) = -\frac{GMm}{r^2(1-\frac{a}{r})^2} \simeq -\frac{GMm}{r^2}\left(1+\frac{2a}{r}\right)$

$f(r) = \underbrace{-\frac{GMm}{r^2}}_{\text{classical}} - \underbrace{2\frac{GMma}{r^3}}_{\text{correction}}$ for $\frac{a}{r} << 1$

c.) $m(\ddot{r} - r\dot{\theta}^2) = f(r)$

$mr^2\dot{\theta} = l$ $\qquad l =$ Angular momentu

$\dot{\theta} = \frac{l}{mr^2}$

Make the standard substitution $u = \frac{1}{r}$

$\dot{r} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt} = \frac{dr}{du}\frac{du}{d\theta}\dot{\theta} = -\frac{\dot{\theta}}{u^2}\frac{du}{d\theta}$

$\dot{\theta} = \frac{l}{m}u^2 \Rightarrow \dot{r} = -\frac{l}{m}\frac{du}{d\theta}$

$\frac{d}{dt} = \frac{d}{d\theta}\dot{\theta} = \frac{lu^2}{m}\frac{d}{d\theta} \Rightarrow \ddot{r} = -\frac{l^2}{m^2}u^2\frac{d^2u}{d\theta^2}$

$f(r) = \cancel{} - GMmu^2 - 2GMmau^3$

$-\frac{l^2}{m}u^2\frac{d^2u}{d\theta^2} - \frac{l^2}{m}u^3 = -GMmu^2 - 2GMma$

$d^2u/d\theta^2 + \left(1 - \frac{GMm^2a}{l^2}\right)u = \frac{GMm^2}{l^2}$

Let $\alpha = \sqrt{1 - \frac{GMm^2}{\ell^2}}$

$$\frac{d^2u}{d\theta^2} + \alpha^2 u = \frac{GMm^2}{\ell^2}$$

For $\frac{GMm^2}{\ell^2} < 1 \quad \alpha^2 > 0$

$$u = A\cos\alpha\theta + \frac{GMm^2}{\ell^2}$$

$$r = \frac{1}{A\cos\alpha\theta + \frac{GMm^2}{\ell^2}} = \frac{\ell^2/GMm^2}{1 + \varepsilon\cos\alpha\theta}$$

This is the form for $r = r(\phi)$, although in point of fact the orbits are not in the plane, but helical.

This problem can be solved more elegantly using the revolving orbits approach. The effective potential of the original problem can be related to that of an equivalent problem with ~~shifted~~ shifted angular momentum etc.

Columbia University
Department of Physics
QUALIFYING EXAMINATION
Monday, January 10, 2005
11:10 AM – 1:10 PM

Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2; Section 2(Electricity etc.) Question 4, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

You may refer to the single note sheet on 8 ½ x 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

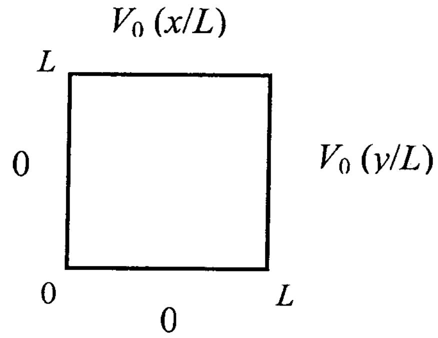
Section 2 – Question 1

A two dimensional box, $0 \le x \le L$, $0 \le y \le L$, is bounded by four conducting plates. The potential boundary conditions are:

$$\phi(0,y) = \phi(x,0) = 0$$

$$\phi(L,y) = V_0 \cdot \frac{y}{L}$$

$$\phi(x,L) = V_0 \cdot \frac{x}{L}$$



a) Determine $\phi(x,y)$ everywhere in the box.

b) Draw qualitatively how equipotential and electric flux lines look like inside the box.

Section 2 – Question 2

A hollow sphere of radius R rotates about one of its diameters with an angular velocity ω. Its surface is electrically charged with a charge density σ.

a) Determine the magnetic moment, $\vec{m}$, of the sphere.

b) Find the magnetic field strength inside and outside the sphere as a function of the distance, r, from its center.

Section 2 – Question 3

The speed of electrons undergoing cyclotron motion can be increased by increasing the magnetic field causing them to move in a circle with time. This is the principle behind a *betatron*. The electrons can be kept in an orbit of constant radius, R, in this process if the magnetic field over the area of the electron's orbit is non-uniform.

Find the relationship between the field at the circumference of the orbit and the average field over the orbit's area that would be required to keep the electron at a constant radius as it is accelerated.

Assume that the electrons start from rest in zero field, and that the apparatus is symmetric about the center of the orbit.

Section 2 – Question 4

Consider a large region of space where the electric field, $\mathbf{E}(x,y,z) = [E_x, E_y, E_z] = [|E|, 0, 0]$, is constant and homogeneous. The effect of gravity can be neglected.

A point particle with mass, m, and positive charge, e, is *accelerated* by the electric field. It moves through the origin at $t_o = 0$. Its velocity vector, $\mathbf{v}$, is known at this point: $\mathbf{v} = [v_o \cos\alpha, 0, v_o \sin\alpha]$, with $v_o << c$ and $\cos\alpha > 0$.

At some later time, t_f, the distance, along the x-axis, between the particle and the origin is L and the particle's speed is still non-relativistic: $v_f << c$.

a) Estimate the total energy emitted, W, in the form of *dipole radiation*, between t_o and t_f, as a function of the variables given above.

b) Describe and interpret the result.

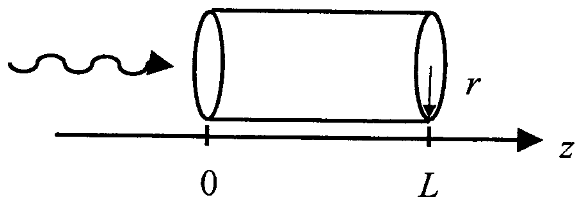
Hint: one can approximate the power radiated by a dipole moment, p, *in non-relativistic situations, by the Larmor formula:*

$$P = \frac{\mu_o \ddot{p}}{6\pi c}$$

Section 2 – Question 5

A plane electromagnetic wave in vacuum is propagating in the positive z-direction. The wave has a frequency ω and its amplitude is being slowly decreased in time. In particular, at $z = 0$ the amplitude is proportional to $(1 - at)$ for time $t = 0$ to $t = 1/a$, with $a/\omega << 1$.

Consider an imaginary cylinder as shown below.



Find the net, average outward energy flow per unit time from the cylinder and show that it equals the rate at which the enclosed energy decreases with time.

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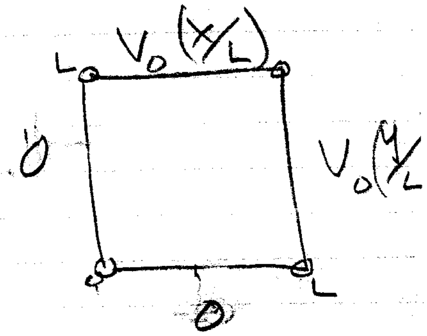
(1) A two dimensional box $0 \le x \le L$, $0 \le y \le$ is bounded by 4 conducting plates. The potential boundary conditions are

$$\phi(0,y) = \phi(x,0) = 0$$

$$\phi(L,y) = V_0 \cdot \frac{y}{L}$$

$$\phi(x,L) = V_0 \cdot \frac{x}{L}$$



a) Determine $\phi(x,y)$ everwhere in the box.

b) Draw qualitatively how equipotentia and electric flux lines look like inside the Box.

E+m solution

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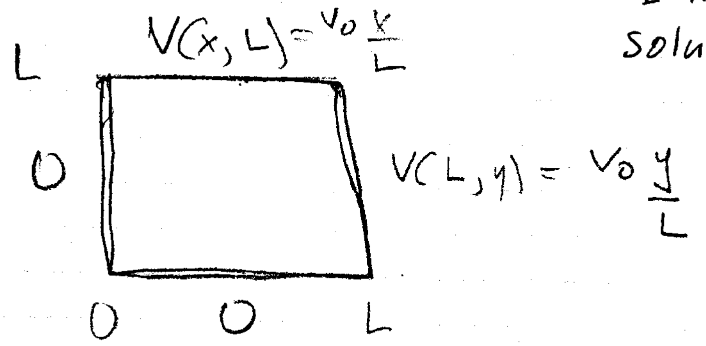

$\Delta \phi(x\,y) = 0$ $\qquad (\partial_x^2 + \partial_y^2)\,\psi_m(x)\,\phi_m(y) = 0$

take $\partial_y^2 \phi_m(y) = -k_m^2 \phi_m$

to satisfy $\phi(x, 0) = 0$ $\qquad \phi_m = \sin k_m x$

$\Rightarrow (\partial_x^2 - k_m^2)\,\psi_m(x) = 0$

$\psi_m(x) = \mathrm{sh}\, k_m x$ $\qquad$ so that $\psi_m(0) = 0$

$\phi(x,y) = \sum_m A_m \,\mathrm{sh}(k_m x) \sin k_m y$

if we take $k_m = \frac{m\pi}{L}$ then boundary at $y = L$ vanish

$\underline{\phi(x, L) = 0}$!

but if we take

$k_m = i\,\frac{m\pi}{m}$ $\qquad$ ($\mathrm{sh}\, ix = i \sin x$)

then $\phi(L, y) = 0$

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E+M
Solution

II-1

thus $\phi = \phi^1 + \phi^2$

when $\phi_1 = \sum_{m=1}^{\infty} A_m \, sh \frac{m\pi x}{L} \, \sin \frac{m\pi y}{L}$

$\phi_2 = \sum_{m=1}^{\infty} B_m \, \sin \frac{m\pi x}{L} \, sh \frac{m\pi y}{L}$

$\phi_1(L, y) \neq 0$ but $\phi_2(L, y) = 0$

$\phi_1(x, L) = 0$ $\qquad \phi_2(x, L) \neq 0$

Fix A_m via $\int_0^L \sin \frac{m\pi y}{L} \sin \frac{m'\pi y}{L} dy$

$= \frac{L}{2} \delta_{mm'}$

using $\langle \sin^2\theta \rangle = \frac{1}{2} \Rightarrow \int \left(\sin^2 \frac{m\pi y}{L}\right) dy$
$= L/2$

$sh(m\pi) A_m = \frac{2}{L} \int_0^L dy \left(\sin \frac{m\pi y}{L} \; V_0\left(\frac{y}{L}\right) \right)$

$= \frac{2V_0}{(m\pi)^2} \int_0^{m\pi} dx (\sin x) \, x$

$f = (-x(\cos x) + \cancel{\sin x})$

$f' = x \sin - \cos + \cos x = x \ldots$

$= \frac{2V_0}{(m\pi)^2} (-(m\pi)) \cos m\pi = (-1)^{m+1} \frac{2V_0}{m\pi}$

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Gyulassy E+M Solution

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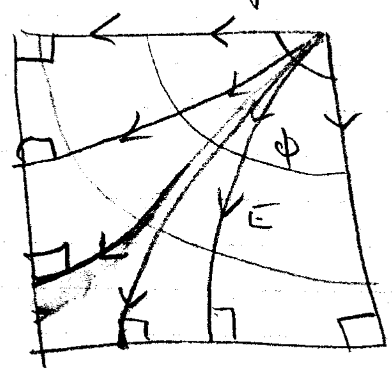
II -1.

$$\boxed{A_m = (-1)^{m+1} \frac{2V_0}{m\pi} \frac{1}{sh(m\pi)}}$$

obviously $B_m = A_m$ by symmetry

$$\boxed{\phi = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{sh(m\pi)} \left(\frac{2V_0}{m\pi}\right) \left\{ sh \frac{m\pi x}{L} \sin\frac{m\pi y}{L} + sh \frac{m\pi y}{L} \sin\frac{m\pi x}{L} \right\}}$$

Draw the qualitatively equipotential and electric lin



Magnetostatics

① A sphere of radius R rotates ~~rotates~~ about one of its diameters with an angular velocity ω. Its surface is electrically charged with a density σ.

ⓐ Determine the magnetic moment $\vec{m}$ of the sphere.

ⓑ Find the magnetic field strength inside and outside the sphere.

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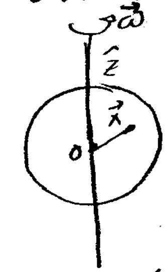
Solution :

(a) The magnetic dipole moment due to a current distribution is given by.



$$m_i = \frac{1}{2c}\int d\vec{x}'\left[\vec{x}'\times \vec{j}(\vec{x}')\right]_i$$

We have to find the $\vec{j}$ due to the rotating sphere.

We can choose the axis of rotation to be the $\hat{z}$ axis.

Since the charge is confined to the surface,

$$\rho(\vec{x}) = \sigma\,\delta(r-R).$$

where $r = |\vec{x}|$.

Check:
$$q = \int dV\,\rho(\vec{x}) = \int_0^\infty dr\,r^2\int d\Omega\,\sigma\,\delta(r-R) = 4\pi R^2\sigma$$

The current is

$$\vec{j}(\vec{x}) = \rho(\vec{x})\,\vec{V}$$

& $\vec{V} = \vec{\omega}\times\vec{x} = \omega\,\hat{z}\times\vec{x}$
$$= \omega\left[x\hat{y} - y\hat{x}\right] = \omega r\sin\theta\left[\cos\phi\,\hat{y} - \sin\phi\,\hat{x}\right]$$

$$\Rightarrow \vec{x}\times\vec{j}(\vec{x}) = \rho(\vec{x})\,\vec{x}\times\vec{V} = \rho(\vec{x})\,\omega r\sin\theta\left[\cos\phi\,\vec{x}\times\hat{y} - \sin\phi\,\vec{x}\times\hat{x}\right]$$

$$= \rho(\vec{x})\,\omega r\sin\theta\left[\cos\phi\{x\hat{z} - z\hat{x}\} - \sin\phi\{-y\hat{z} + z\hat{y}\}\right]$$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$$= \rho(\vec{x})\,\omega\sin\theta\,r\Big[\cancel{(-r\cos\theta\cos\phi + r\sin\theta\sin^2\phi)\hat{z}}$$
$$r(\sin\theta\cos^2\phi + \sin\theta\sin^2\phi)\hat{z}$$
$$- r\cos\theta\cos\phi\,\hat{x} - r\cos\theta$$

$$= \omega\rho(\vec{x})\,r\sin\theta\left[r\sin\theta\,\hat{z} - r\cos\theta\cos\phi\,\hat{x} - r\cos\theta\sin\phi\,\hat{y}\right]$$

Now $\int_0^{2\pi} d\phi \cos\phi = \int_0^{2\pi} d\phi \sin\phi = 0$.

$$\Rightarrow \int d\vec{x}\,[\vec{x}\times\vec{j}(\vec{x})] = \omega\hat{z}\int d\vec{x}\,\rho(\vec{x})\,r^2\sin^2\theta$$

$$= \omega\sigma\hat{z}\int_0^\infty dr\,r^2 r^2 \underbrace{\int_0^\pi d\theta\,\sin\theta\,\sin^2\theta}_{=\int_{-1}^{1} d\eta\,(1-\eta^2)}\,\delta(r-R)\underbrace{\int_0^{2\pi} d\phi}_{=2\pi}$$

where $\eta \equiv \cos\theta$

$$= 2 - \frac{2}{3} = \frac{4}{3}.$$

$$\Rightarrow \int d\vec{x}\,[\vec{x}\times\vec{j}(\vec{x})] = \omega\sigma\hat{z}\,R^{4}\cdot\frac{4}{3}\cdot 2\pi$$

$$= \frac{8\pi}{3}\,\omega\sigma R^4\hat{z}$$

$$\Rightarrow \vec{m} = \frac{4\pi\omega\sigma}{3c}R^4\hat{z} = \frac{\omega\sigma}{c}VR\hat{z} = \frac{\omega Q}{3c}R^2\hat{z} = \frac{QR^2}{3c}\vec{\omega}.$$

where V = volume of the sphere.

$$\boxed{\vec{m} = \frac{QR^2}{3c}\vec{\omega}}$$

$$= \frac{4\pi R^4\sigma}{3c}\vec{\omega}$$

$$\boxed{\vec{m} = \frac{4\pi}{3c}R^4\sigma\,\vec{\omega}}$$

(b) Since there are no currents either inside or outside the sphere, we can describe the magnetic field inside/outside the sphere in terms of magnetic scalar potentials.

If $\vec{B}_1$ is the field inside the sphere & $\vec{B}_2$ that outside,

$$\left.\begin{aligned} \vec{B}_1 &= -\vec{\nabla}\varphi_1 \\ \vec{B}_2 &= -\vec{\nabla}\varphi_2 \end{aligned}\right\}.$$

$\vec{B}_1$ & $\vec{B}_2$ satisfy boundary conditions at $r=R$:

$$\left.\begin{aligned} B_{1n} &= B_{2n} \\ \& \quad \hat{n}\times(\vec{B}_2-\vec{B}_1) &= \vec{j}/c \end{aligned}\right\}$$

Recall: $\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

Now $\vec{j} = \sigma \omega R \sin\theta\,\hat{\phi}$ (from part ⓐ)

So $\frac{\sigma\omega}{c} R \sin\theta = \left[\hat{n}\times(\vec{\nabla}\varphi_1 - \vec{\nabla}\varphi_2)\big|_{r=R}\right]_\phi$

But $\hat{n} = \hat{r}$

$$\Rightarrow \boxed{\frac{\sigma\omega}{c} R \sin\theta = \frac{1}{R}\left(\frac{\partial\varphi_1}{\partial\theta} - \frac{\partial\varphi_2}{\partial\theta}\right)\Big|_R} \quad \text{--- ⓐ}$$

$$\& \quad \boxed{\frac{\partial\varphi_1}{\partial r}\Big|_R = \frac{\partial\varphi_2}{\partial r}\Big|_R} \quad \text{--- ⓑ}$$

φ_1 & φ_2 satisfy $\nabla^2 \varphi_{1,2} = 0$ (as $\vec{\nabla}\cdot\vec{B} = 0$)

Given the obvious azimuthal symmetry, we can write

$$\varphi_1(r,\theta) = \sum_{l=0}^{\infty} A_l\, r^l\, P_l(\cos\theta) \quad \text{(Has to be regular at } r = \text{)}$$

$$\& \quad \varphi_2(r,\theta) = \sum_{l=0}^{\infty} B_l\, \frac{1}{r^{l+1}}\, P_l(\cos\theta) \quad \text{(Has to die out at } r \to \text{)}$$

Now $P_1(\cos\theta) = \cos\theta$ & $\dfrac{dP_1}{d\theta} = -\sin\theta$.

From (a) $\Rightarrow$ $\underline{A_l, B_l \text{ for } l \neq 1 \text{ are zero.}}$

$$\Rightarrow \frac{\sigma\omega R}{c}\sin\theta = \frac{1}{R}\left[A_1 R(-\sin\theta) - B_1\frac{1}{R^2}(-\sin\theta)\right]$$

$$\Rightarrow \boxed{A_1 - \frac{1}{R^3}B_1 = -\frac{\sigma\omega R}{c}} \quad \text{--- (c)}$$

from (B), $$A_1\cos\theta + \frac{2B_1\cos\theta}{R^3} = 0$$

$$\Rightarrow \boxed{B_1 = -\frac{R^3}{2}A_1} \quad \text{--- (d)}$$

(c), (d) $\Rightarrow$ $$\frac{3A_1}{2} = -\frac{\sigma\omega}{c}R$$

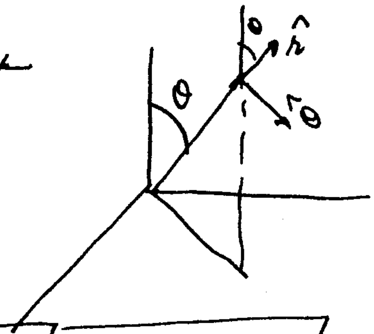
$$\Rightarrow \boxed{A_1 = -\frac{2}{3c}\sigma\omega R} \quad \text{--- (e)}$$

$$\Rightarrow \boxed{B_1 = \frac{\sigma\omega}{3c}R^4} \quad \text{--- (f)}$$

So,
$$\left.\begin{aligned} \varphi_1(r,\theta) &= -\frac{2}{3c}\sigma\omega R\, r\cos\theta \\ \varphi_2(r,\theta) &= \frac{\sigma\omega}{3c}R^4\frac{1}{r^2}\cos\theta \end{aligned}\right\} \quad \text{--- (g)}$$

So, $B_r = -\frac{\partial \psi_1}{\partial r} = \frac{2}{3c}\sigma\omega R\cos\theta$

& $B_{1\theta} = -\frac{1}{r}\frac{\partial}{\partial\theta}\psi_1 = -\frac{2}{3c}\sigma\omega R\sin\theta$.



$\cancel{\vec B_1 = \hat r B_{1r} + \hat\theta B_{1\theta} = (\hat x \sin\theta\cos\phi +}$

$\Rightarrow \vec B_1 = \frac{2}{3c}\sigma\omega R\,(\underbrace{\hat r\cos\theta - \hat\theta\sin\theta}_{=\hat z})$

$\Rightarrow \boxed{\vec B_1 = \frac{2}{3c}\sigma\omega R\,\hat z = \frac{2\sigma}{3c}R\vec\omega = \frac{1}{2\pi R^3}\vec m}$

$B_{2r} = -\frac{\partial\psi_2}{\partial r} = \frac{2\sigma\omega R^4}{3}\frac{1}{r^3}\cos\theta$.

$B_{2\theta} = -\frac{1}{r}\frac{\partial\psi_2}{\partial\theta} = \frac{\sigma\omega}{3}R^4\frac{1}{r^3}\sin\theta$.

Now $\vec m\cdot\vec r = \frac{4\pi\sigma R^4\omega}{3c}\,\hat z\cdot\vec r = \frac{4\pi\sigma}{3c}R^4\omega r\cos\theta$

Let $\vec\Sigma \equiv \frac{3(\vec m\cdot\vec r)\vec r}{4\pi r^5} - \frac{\vec m}{4\pi r^3} = \frac{\sigma\omega R^4\cos\theta}{cr^3}\hat r - \frac{\sigma\omega R^4}{3cr^3}\hat z$

Then $\Sigma_r = \frac{\sigma\omega R^4}{cr^3}\cos\theta - \frac{\sigma\omega R^4}{3cr^3}\cos\theta = \frac{2}{3}\frac{\sigma\omega R^4}{cr^3}\cos\theta$

[Note $\hat z\cdot\hat r = \cos\theta$] $= \cancel{\tfrac{1}{c}}\,B_{2r}$.

& $\Sigma_\theta = -\frac{\sigma\omega R^4}{3cr^3}\underbrace{\hat z\cdot\hat\theta}_{=-\sin\theta} = \frac{\sigma\omega R^4}{3cr^3}\sin\theta = \cancel{\tfrac{1}{c}}\,B_{2\theta}$

$\Rightarrow \boxed{\vec B_2 = \cancel{c}\,\vec\Sigma = \frac{3\cancel{c}(\vec m\cdot\vec r)\vec r}{4\pi r^5} - \frac{\cancel{c}\,\vec m}{4\pi r^3}}$. precis the field of a magnetic dip

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E+M (1)

II - 4

Radiation related problem

Consider a sufficiently large region in vacuum around the origin, where the electric field ($\vec{\mathbf{E}}(x, y, z) = [E_x, E_y, E_z] = [|E|, 0, 0]$) is constant and homogenous (i.e. far beyond the volume of interest). The effect of gravity can be neglected.

A point particle with mass (m) and positive charge (e) is *accelerated* by the electric field. It moves through the origin at $t_\circ$=0. Its velocity vector $\vec{\mathbf{v}}$ is known at this point: $|\vec{\mathbf{v}}| = v_\circ \ll c$ and $v_y = 0$, where c is the speed of the light. The angle between $\hat{v}$ and $\hat{x}$ is α; $cos(\alpha) > 0$.

The distance *along the x-axis* between the origin and the point particle will be L at some later time (t_f).

Estimate the total energy emitted (W) in the form of *dipole radiation* between $t_\circ$ and t_f as a function of $\left(\vec{\mathbf{E}},\ v_\circ,\ \alpha,\ L,\ m\right)$. α and L is chosen that the charge never leaves the region of interest where the field is constant and homogenous and that the speed of the particle at t_f is still non-relativistic ($v_f \ll c$). Describe and interpret the result.

(*Hint: Remember, one can approximate the (dipole) radiated power with the Larmor formula (e.g. $P \cong \frac{\mu_\circ \ddot{p}}{6\pi c}$, where p is the dipole moment) for non-relativistic situations.*)

Summary of solution:

One way to estimate this is:

The equation of motion is $m\ddot{\vec{\mathbf{r}}} = e\vec{\mathbf{E}}$. Since $\ddot{\vec{\mathbf{p}}} = e\vec{\mathbf{a}}$ in this case, the radiated power can be approximated as $P \cong \frac{\mu_\circ}{6\pi c}\frac{e^2E^2}{m^2}$ according to the Larmor formula.

The x-projection of the position of the particle at time t_f is $L = \frac{1}{2}\left(\frac{eE}{m}\right)t_f^2 + v_\circ t_f$.

Therefore the total flight time is $t_f = \frac{-mv_\circ cos(\alpha)+\sqrt{m^2v_\circ^2 cos^2(\alpha)+2eEmL}}{eE}$.

Consequently the total energy emitted in the form of dipole radiation between $t_\circ$ and t_f can be estimated as:

$W \cong \frac{\mu_\circ}{6\pi c}\frac{eEv_\circ}{m}\left(\sqrt{cos^2(\alpha)+\frac{2eEL}{mv_\circ^2}} - cos(\alpha)\right)$

E+M question. December 2004. Robert Mawhinney II-5

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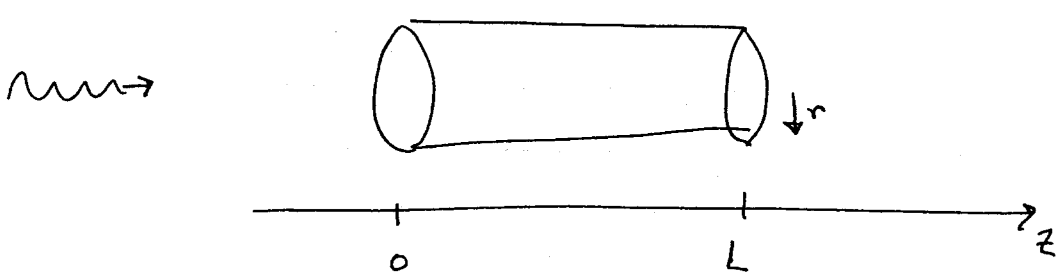
A plane electromagnetic wave in vacuum is propagating in the positive z direction. The wave has frequency ω and its amplitude is being slowly decreased in time. In particular, at z=0 the amplitude is proportional to $(1 - at)$ for time 0 to time $1/a$ and $a/\omega \ll 1$.

Consider an imaginary cylinder as shown



Find the net average outward energy flow per unit time from the cylinder and show that it equals the rate at which the enclosed energy decreases with time.

Solution - page 1

We need the value of $\vec{E}\times\vec{B}$, the Poynting vector, integrated over the surface.

During the time the amplitude is varying, the electric field at $z=0$ is

$$\vec{E} = \vec{E}_0 (1-at)\, e^{i\omega t}$$

The field at a point z at time t is the same as the field at point 0 at time $t - z/c$, which gives

$$\vec{E} = \vec{E}_0 \left(1 - a\left(t - z/c\right)\right) e^{i\omega\left(t - z/c\right)}$$

$\vec{E}\times\vec{B}$ points in the direction of propagation, so we only need the surface integral over the ends of the cylinder. $B = cE\epsilon_0$ and since a/ω is small averages over an oscillation period yield $1/2$ we have,

$$\int (\vec{E}\times\vec{B})\cdot dA = \frac{1}{2}\epsilon_0 c E_0^2 (\pi r^2)\left[\left(1 - at + aL/c\right)^2 - (1-at)^2\right]$$

contribution from right end

contribution from left end

$$= \frac{1}{2}\epsilon_0 c\, E_0^2 (\pi r^2)\left[2(1-at)\frac{aL}{c} + \left(\frac{aL}{c}\right)^2\right]$$

The energy density is $\frac{\epsilon_0 E^2 + B^2/\mu_0}{2} = \epsilon_0 E^2$ for a plane wave. Averaging over a period gives a factor of ½, yielding an average energy of

$$W = \int_0^L \frac{\epsilon_0 E_0^2}{2}\left(1 - at + \frac{az}{c}\right)^2 dz\,(\pi r^2)$$

$$= \frac{\epsilon_0 E_0^2 \pi r^2}{2} \int_0^L dz \left[(1-at)^2 + 2(1-at)\frac{a}{c}z + \frac{a^2 z^2}{c}\right]$$

$$= \frac{\epsilon_0 E_0^2 \pi r^2}{2} \left[(1-at)^2 L + (1-at)\frac{a}{c}L^2 + \frac{a^2 L^3}{3c}\right]$$

$$\frac{dW}{dt} = \frac{\epsilon_0 E_0^2 \pi r^2}{2}\left[2(1-at)(-aL) - \frac{a^2 L^2}{c}\right] = -\int (\vec{E}\times\vec{B})\cdot d\vec{A}$$