Columbia University
Department of Physics
QUALIFYING EXAMINATION
January 16, 2004
9:00 AM – 11:00 AM

General Physics (Part I)
Section 5. Thermodynamics, Statistical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing additional problems). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2; Section 5(General Physics) Question 7, etc.)

Do **NOT** write your name on your answer booklets. Instead clearly indicate your **Exam Letter Code**

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Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good luck!!

Problem 1

An ideal gas of N non-interacting fermions in a volume V is in thermal equilibrium at temperature T.

Denote the mean occupation number of a state with energy E as $<n(E)>$, and the corresponding number fluctuations as

$$<\Delta n(E)^2> = <n^2(E)> - <n(E)>^2 .$$

Find the value of E that maximizes this quantity, compute the maximal value, and comment on the physical interpretation.

Problem 2

A spherical black body of radius R and temperature T is surrounded by a larger co-centered reflecting spherical shell of radius R_R.

a) What is the thermal equilibrium energy density per unit angular frequency interval, $u(\omega, T)$, of the electromagnetic radiation in the space, $R < r < R_R$. Hint: you might want to use $\dfrac{\sum\limits_{n=0}^{\infty} n e^{-nx}}{\sum\limits_{n=0}^{\infty} e^{-nx}} = \dfrac{1}{e^x - 1}$ to get $u(\omega, T)$.

b) Suppose the $R < r < R_R$ region is filled with a non-absorbing medium with index of refraction n. What is the new electromagnetic energy density $u(\omega, T)$?

c) A thin layer reflecting a fraction $\hat{f}$ of any electromagnetic radiation incident upon it is put onto the surface of the blackbody sphere in a). How is $u(\omega, T)$ changed?

d) Suppose the region $R < r < R_R$ is filled with a liquid whose sound speed is C_s. What is the equilibrium energy density per unit frequency, $u_s(\omega, T)$, of the sound waves there (assume $\dfrac{\hbar c_s}{k_B T} \gg$ inter-atomic spacing)?

e) A very tiny hole of area A is opened in the larger reflecting spherical shell and is covered by the same reflecting material used in part c) to cover the central blackbody. What is the power in escaping radiation?

Problem 3

Consider an ideal gas of N particles of mass m confined to a cubic box of volume V ($V=L^3$). The box is in a uniform gravitational field $-g\hat{y}$ and is at equilibrium at temperature T. Assume that the potential energy of a particle is $U(y)=mgy$ where y is the vertical coordinate inside the box ($-L/2 < y < L/2$).

a) Write down the partition function of the system using momentum and coordinates.

b) Find the energy of the system.

c) Find the heat capacity of the system.

d) Find the density and pressure distribution as a function of the height y.
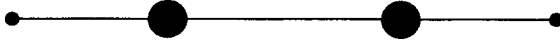

Problem 4

Consider a system of non-interacting particles of spin ½. Each particle has mass m, and is constrained to move with non-relativistic momentum on the very flat surface of superfluid liquid He kept at very low temperature $T\to0$. The components of the particles' momenta are restricted to the (x,y) plane that defines the surface of the liquid. The interactions between particles and the liquid He substrate are negligible. The area of the surface is $A=L^2$, where L is the length of the surface along the x and y-directions. The density of particles per unit area is n.

a) Assume cyclic boundary conditions for the wavefunctions of momentum and energy to obtain the allowed energy states. [In cyclic boundary conditions the wavefunctions at position (x,y) are identical to those at position (x+L, y+L)].

b) Use the results in a) to obtain an expression for $g(E)$ such that $g(E)dE$ gives the number of energy states of the particles in the range $[E,E+dE]$.

c) Consider the limit T=0. Obtain the energy difference between the lowest and highest energy states occupied by the particles.

d) What is the average energy per particle at T=0?

e) The temperature is raised slightly so that ΔT is much smaller than the average energy per particle. Describe in words the changes that occur in the system. What is the expected temperature dependence of the change in the total energy of the system?
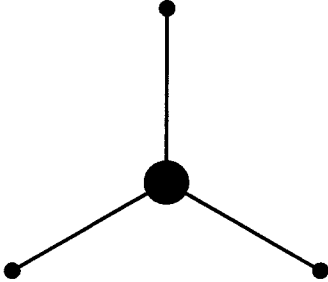
Problem 5

Calculate the classical specific heat, C_V, of the ideal gas consisting of N

a) Linear molecules as shown:

b) Planar molecules as shown:

Problem 6

A long (length d). solid, and well-insulated rod has one end attached to a cold temperature reservoir so that the entire rod is initially at the reservoir temperature. The rod is characterized by a known specific heat C, density ρ, and thermal conductivity κ. If the other end of the rod is suddenly connected to a high temperature reservoir a temperature ΔT higher, then

a) At $t=0$, find the heat intensity (power per unit area) that flows into the low temperature reservoir.
b) At thermal equilibrium, find the heat intensity that flows into the low temperature reservoir.
c) **Estimate** the time it takes for this equilibrium to be established.

Answers

a)

$$u(\omega)\,d\omega = \frac{\hbar\omega}{e^{\hbar\omega/k_BT} - 1}\;\frac{2\cdot 4\pi k^2\,dk}{(2\pi)^3}$$

$$\text{with}\quad k = \frac{\omega}{c}$$

b)   as above with $k = \dfrac{n\omega}{c}$

c) T is unchanged

d)   as in a) with $c \to c_s$
and   (2) $\to$ (1)   [one mode]

e)

$$u(\omega)\,d\omega = \frac{\hbar\omega^3\,d\omega}{e^{\hbar\omega/k_BT} - 1}\;\frac{4\pi}{(2\pi)^3}\left\{\frac{1}{c_1^3} + \frac{1}{c_2^3}\right\}$$

note that in a 2-D layer there is only a single kind of shear waves (+ the usual compressional wave)

f) $A(1-\hat{f})\dfrac{c}{4}\,u(\omega,T)$

Thermo /Stat   Canonical Ensemble .

(a) $H = \frac{P^2}{2m} + mgz$, $\beta = \frac{1}{k_B T}$

$$Z_1 = \left[\frac{1}{h^3} \int d\vec{p}^3 \int d\vec{r}^3 \, e^{-\beta H} \right] = \left[\frac{1}{h^3} \left[\int_{-\infty}^{\infty} dp \, e^{-\frac{\beta p^2}{2m}} \right]^3 L^2 \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{-\beta mgz} dz \right]$$

$$= \left[\frac{2\pi m}{\beta h^2} \right] L^2 \frac{2}{\beta mg} \sinh \left[\frac{\beta mg L}{2} \right] .$$

$Z_{tot} = Z_1^N .$ $\beta = \frac{1}{k_B T}$

(b)

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z_{tot}$$

$$= \frac{3}{2} \frac{N}{\beta} - N \frac{\partial}{\partial \beta} \left[-\ln \beta + \log \left\{ \sinh \left(\frac{\beta mg L}{2} \right) \right\} \right]$$

$$= \frac{5}{2} N k_B T - \frac{N mg L}{2} \coth \left[\frac{mg L}{2 k_B T} \right]$$

(c)

$$C = \frac{1}{N} \frac{\partial}{\partial T} \langle E \rangle = \frac{5}{2} k_B - \frac{mg L}{2} \left(\frac{g m L}{2 k_B} \frac{-1}{T^2} \right) \coth' \frac{mg L}{2 k_B T}$$

$\underset{= -\sinh^2}{\underbrace{}}$

$$= k_B \left[\frac{5}{2} - \left(\frac{mg L}{2 k_B T} \right)^2 \sinh^{-2} \left(\frac{mg L}{2 k_B T} \right) \right]$$

(d)

$$n_{(z)} = \frac{N e^{-mgz \cdot \beta}}{L^2 \int_{-\frac{L}{2}}^{\frac{L}{2}} dz \, e^{-mgz\beta}} = N mg\beta \frac{e^{-mgz\beta}}{2\sinh\left(\frac{mg\beta L}{2}\right)}$$

$$= \frac{N}{L^3}\left(\frac{mgL}{2k_BT}\right) \frac{e^{-mgz/k_BT}}{\sinh\left[\frac{mgL}{2k_BT}\right]} \quad //$$

$$P_{(z)} = k_BT \cdot n_{(z)}$$

$$= \frac{N}{L^3}\left(\frac{mgL}{2}\right) \frac{e^{-mgz/k_BT}}{\sinh\left[\frac{mgL}{2k_BT}\right]} \quad //$$

Solution

General: particle statistics / thermal

(a)

momentum : $\vec{p} = \hbar \vec{k}$

The wave functions are

$$\psi = A \, e^{i\vec{k}\cdot\vec{r}} \, e^{-i\omega t}$$

where $\quad E = \hbar \omega$

In cyclic boundary conditions

$$e^{ik_x L} = e^{ik_y L} = 1$$

$$k_x = m_x \frac{2\pi}{L} \quad ; \quad k_y = m_y \frac{2\pi}{L}$$

where $\quad m_x, m_y$ are integers $\quad |m_x| \leq \infty$

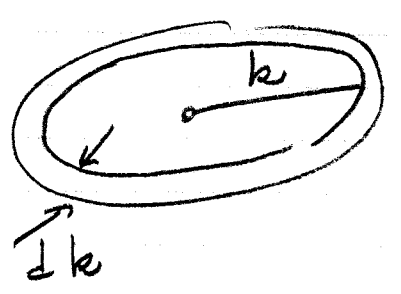$$|m_y| \leq \infty$$

The energies are

$$E = \frac{\hbar^2 k_x^2}{2m} + \frac{\hbar^2 k_y^2}{2m} \, \xi$$

$$E(m_x, m_y) = \frac{2\pi^2 \hbar^2}{m L^2}\left(m_x^2 + m_y^2\right)$$

(b)

In wavevector space the area is

$$\pi k^2 = A_k$$

where
$$k^2 = k_x^2 + k_y^2$$

$$dA_k = 2\pi k \, dk$$

they are assigned to each point is

$$\left(\frac{2\pi}{L}\right)^2 = \frac{4\pi^2}{L^2}$$

The number of states in dA_k is

$$dN = \frac{L^2}{4\pi^2} \, 2^{\nearrow \text{spin degeneracy}} \, 2\pi k \, dk$$

and per unit area we have

$$dn = \frac{1}{\pi}(k \, dk) = g(E) \, dE$$

$$(k \, dk) = \frac{m}{\hbar^2} \, dE$$

$$dn = \frac{m}{\pi \hbar^2} \, dE$$

$$g(E) = \frac{m}{\pi \hbar^2} \qquad E \geq 0$$

(c)

The particles are fermions so that

$$n = \int_0^{E_{max}} g(E) \, dE$$

$$n = \frac{m}{\pi \hbar^2} E_{max}$$
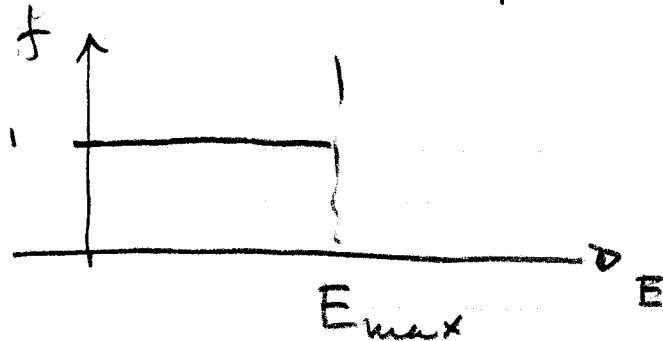
$$E_{max} = \frac{\pi \hbar^2 n}{m}$$

(d)

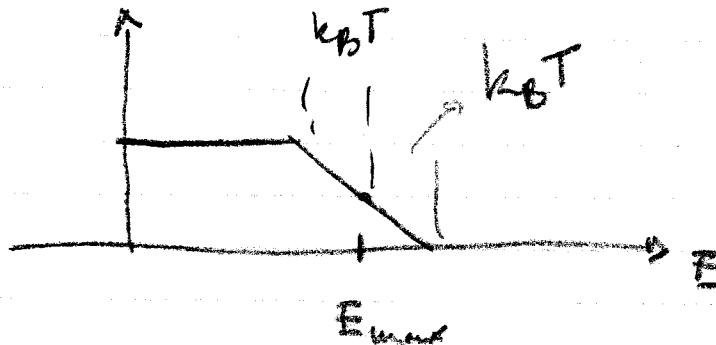$$\langle E \rangle = \int_0^{E_{max}} E \, g(E) \, dE = \frac{m}{\pi \hbar^2} \frac{E_{max}^2}{2}$$

$$= \frac{1}{2} \frac{m}{\pi \hbar^2} E_{max} E_{max} = \frac{1}{2} \frac{m}{\pi \hbar^2} \frac{\pi \hbar^2}{m} n \, E_{max}$$

$$= \frac{1}{2} n \, E_{max}$$

(e) At $T = 0$ the prob. to find a particle at energy E is
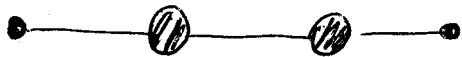

$$E_{max}$$

at ΔT

$$E_{max}$$

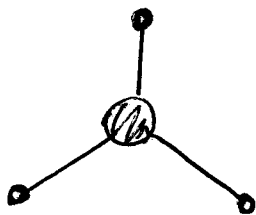The number of particles $\Delta n \sim (k_B T)$ are excited above E_{max}

$$\Delta E \sim k_B T \, \Delta n \sim (k_B T)^2$$

#2. Calculate classical specific heat (C_V)
of the ideal gas consisting of N

a) Linear molecules as shown

b) Planar molecules as shown

Solution:

$$C_V = \frac{N}{2}\left(n_{cm} + n_r + 2n_v\right)$$

where n_{cm} is the number degrees of freedom associated with the center of mass, n_r is the number of rotational degrees of freedom and n_v is the number of vibrational degrees of freedom

a) $n_{cm} = 3$; $n_r = 2$; $n_v = 7$; $C_V = \frac{19N}{2}$;

b) $n_{cm} = 3$; $n_r = 3$; $n_v = 6$; $C_V = 9N$

Solution — heat transport problem:

F. Sciulli
Qual 2004

Using freshman physics
definition k:
$$\frac{Q}{t} = kA\frac{\Delta T}{d} \overset{1D}{\Rightarrow} I \equiv \frac{1}{A}\frac{dQ}{dt} = k\frac{dT}{dx}$$

definition C
$$Q = mC\Delta T \overset{1D}{\Rightarrow} \frac{dI}{dx} = \rho C\frac{dT}{dt}$$

$$\left.\right\} \quad \frac{d^2T}{dx^2} - \frac{\rho C}{k}\frac{dT}{dt} = 0$$

(diffusion equation)

1. At $t=0$, no heat has yet reached low temp reservoir
$$I_{low}\big|_{t=0} = 0$$

2. At $t=\infty$, equilibrium established
$$I_{low}\big|_{t=\infty} = k\frac{\Delta T}{d}$$

3. An $\underline{\text{estimate}}$ of the average $\left\langle\frac{d^2T}{dx^2}\right\rangle_{t=0} = \frac{1}{d}\left[\left\langle\frac{dT}{dx}\right\rangle_{hi} - \left\langle\frac{dT}{dx}\right\rangle_{lo}\right]$

$$= \frac{1}{d}\left[\frac{\Delta T}{d} - 0\right]$$

$$= \frac{1}{d^2}\Delta T$$

$\Delta t =$ time to reach equilibrium

$$\frac{\Delta T}{\Delta t} = \frac{k}{\rho C}\left\langle\frac{d^2T}{dx^2}\right\rangle = \frac{k}{\rho C}\frac{\Delta T}{d^2}$$

$$\Delta t \approx \frac{\rho C}{k}d^2$$

(Also can get from dimensional analysis!)

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11:10 AM – 1:10 PM

General Physics (Part II)
Section 6.

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Good luck!!

Problem 1

A panel of US physicists and astronomers has identified a list of eleven fundamental questions about the nature of the universe that will require the combined skills of particle physicists and astrophysicists to answer. The questions are in "From quarks to the cosmos", the first report from the committee on the physics of the universe set up by the National Academy of Sciences. Several of the questions were discussed by speakers in the Fall, 2003 Colloquium series. The eleven questions are:

- What is dark matter?
- What are the masses of the neutrinos, and how have they shaped the evolution of the universe?
- Are there additional spacetime dimensions?
- What is the nature of the dark energy?
- Are protons unstable?
- How did the Universe begin?
- Did Einstein have the last word on gravity?
- How do cosmic accelerators work and what are they accelerating?
- Are there new states of matter at exceedingly high density and temperature?
- Is a new theory of matter and light needed at the highest energies?
- How were the elements from iron to uranium made?

For full credit: Choose one of the questions above which was also discussed by a colloquium speaker. Briefly describe the topic of the talk. Explain the larger mystery behind the question and what types of future experiments will address the issue.

For 80% credit: You need not refer to a specific colloquium. Choose one of the questions above. Explain the larger mystery behind the question and what types of future experiments will address the issue.

NOTE: You will only be graded on content. You will not be graded on grammar, spelling or composition.

Problem 2
In an Ar-ion laser, green light (515 nm) is emitted by a transition of Ar^+ ions in a discharge. The radiative lifetime of the relevant transition is $\tau_{sp} = 10$ ns, and its measured line-width is $\Delta v = 3$ GHz. The discharge has an effective temperature of T=1000 K and is at a pressure of 0.1 atmospheres. The laser cavity consists of parallel mirrors separated by 1 m.

a) Estimate the contribution to the line-width from the radiative decay (the natural line-width).
b) Estimate the contribution to the line-width from Doppler broadening.
c) Estimate the contribution to the line-width from pressure broadening (collisions).
d) Estimate in how many different longitudinal modes can the laser oscillate.
e) If the laser is operated in pulsed mode, estimate the duration of the shortest pulse it can support.
f) If the average power if the laser is 10 W, what would be the peak power achievable for the laser operating in pulsed mode in which a single pulse travels back and forth in the cavity? What would be the peak intensity (irradiance) of such a pulse focused to a diffraction-limited spot?
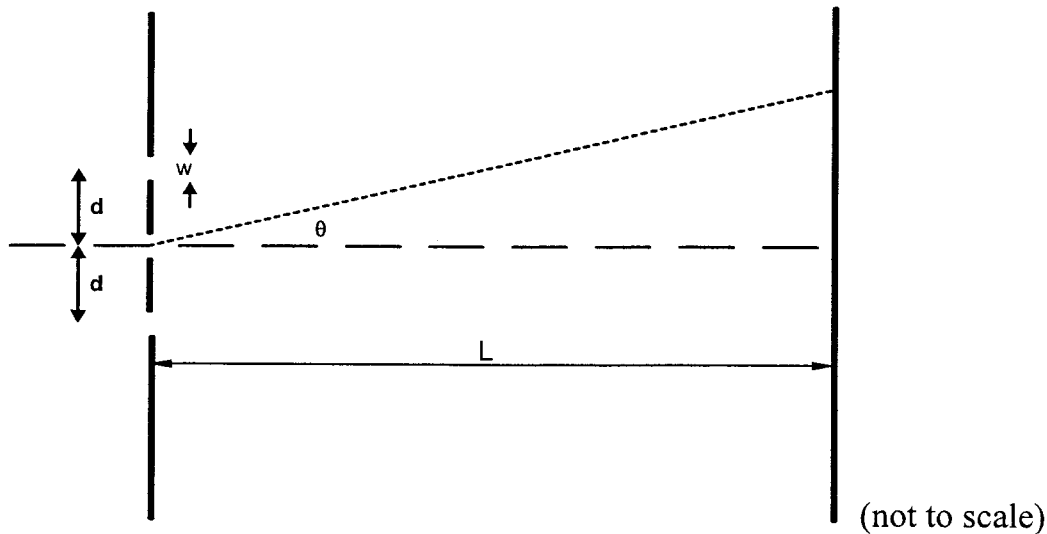
Problem 3
A freshman comes to you and is puzzled by the following argument regarding the pressure in a U-tube containing water. Her physics professor has told her that the pressure at points 1 and 2 are the same and that pressure at a point in a fluid is due to the weight of the fluid above it. The student argues that the pressure at point 1 and point 2 must be different because they have different amounts of fluid above them. Provide a simple argument that explains why the student's reasoning is incorrect.
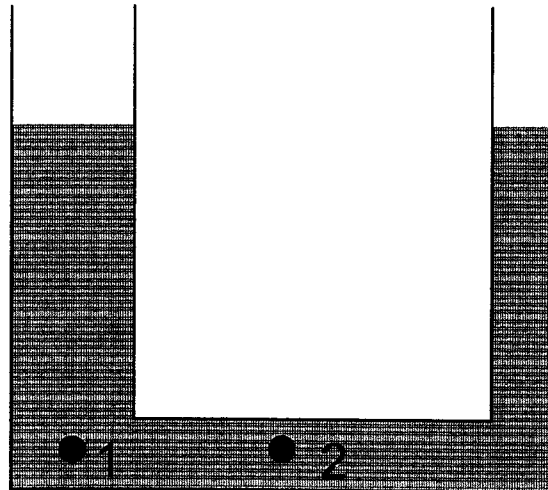
Problem 4

A beam of laser light of intensity I_0 and wavelength λ shines on a perfect absorber with three long slits of width w spaced by a distance d. A flat screen is located a distance L from the absorber.

a) Under what conditions (express as an inequality involving λ) does the finite size of the slits have negligible influence on the diffraction pattern observed on the screen?

b) Find an approximate expression for the angular dependence of the intensity of light observed on the screen for $L/d >> 1$ under the conditions in part a).

c) Describe qualitatively how the shape of the central ($\theta=0$) peak in the diffraction pattern changes when the finite size of the slits is considered.

d) Find an approximate expression for the angular dependence of the intensity of light observed on the screen for $L/d >> 1$ and $L/w >> 1$ taking into account the finite width of the slits.


(not to scale)

Problem 5

Consider a steady rain of particles falling freely from infinity onto a gravitating sphere of mass M and radius R. The rain is spherically symmetric (radial) and its velocity at infinity is zero. The rate of the rain dM/dt [g/s] is given. The falling particles collide inelastically with the sphere and heat it. M grows very slowly as the rain material is accumulated on the sphere, so that the process is quasi-steady.

a) Find the emitted power L [erg/s] and temperature T of the sphere, assuming that it emits as a blackbody.

b) Suppose the sphere rotates with angular velocity omega. What is the spin-down rate $d\Omega/dt$? In the calculation assume that the sphere with the accumulated rain material is a solid body of constant density ρ.

c) What is the luminosity L of the rotating sphere?

Problem 6

A simple model of an insect eye consists of a spherical array of closely packed conical receptors, each optically isolated from its neighbors. The radius of the spherical array is R. Each cone has a small lens, diameter D, which focuses parallel light onto a photosensitive cell within the cone at a distance r from the center.

a) The light and the angular resolution for each element is limited by the diffraction associated with aperture D. Show that if the insect's eye is simultaneously to achieve maximum sensitivity to light and maximum angular resolving power, the photosensitive cell should lie at r = 2R/3 from the center.

b) Estimate the diameter of a single lens in the optimized eye and the minimum resolvable angle between two point sources of green light ($\lambda = 500$ nm) if R = 1 mm.
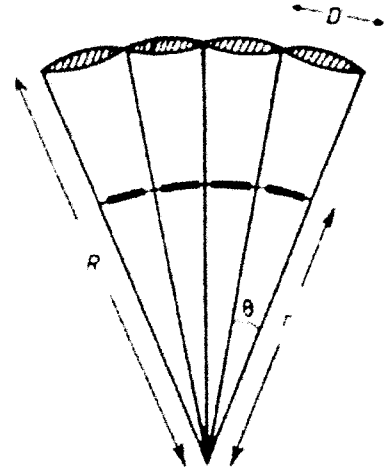
Hi Lalla.

Conrod
Section 6
Question #__1__

FyI: Grading:

Max = 15 if a colloq is listed
Max = 12 if no colloq listed

Top score has:

$\frac{1}{3}$ 1) Explanation of issue

$\frac{1}{3}$ 2) Reference to theories that might explain

$\frac{1}{3}$ 3) Experiments listed which have already or might address question

Section 6,   Question # 2

(Heinz) - Problem 2

(a) $\Delta \nu_{op} = \dfrac{1}{2\pi \tau_{sp}} = 16 \text{ MHz}$

(b) $\Delta \nu_D \approx 2\dfrac{\bar{v}_z}{c}\nu = \dfrac{2\bar{v}_z}{\lambda} \approx \dfrac{2\sqrt{k_B T/M}}{\lambda} = \dfrac{2 \times 680 \, m/s}{511 \, nm} = 2.7 \text{ GHz}$

(c) $\Delta \nu_{coll} = \dfrac{f_{coll}}{\pi}$

$f_{coll} \approx \bar{v} \, \pi d^2 n = \sqrt{3} \, \bar{v}_z \, \pi d^2 n$, with $d \approx 1\text{Å}$ atomic diameter

$\approx 110 \text{ MHz}$

$n = 6 \times 10^{23}/210 \, \ell$

$\Delta \nu_{coll} \approx 35 \text{ MHz}$

$= 3 \times 10^{24} \, m^{-3}$

(d) $\Delta \nu_{mode} = \dfrac{c}{2L} = 150 \text{ MHz}$

Number of accessible modes $\sim \dfrac{\Delta \nu}{\Delta \nu_{mode}} = 20$ (2x if you include polarization)

(e) By the time-frequency uncertainty relation

$\tau \geq (4\pi \Delta \nu)^{-1} = 26 \, ps$

(f) The longest separation possible between the pulses is one cavity roundtrip of $2m/3 \times 10^8 \, m/s = 6.7 \, ns$ Then

$\bar{P} = P_{peak} \dfrac{25 \, ps}{6.7 \, ns} \Rightarrow P_{peak} = \bar{P} \dfrac{6.7 \, ns}{25 \, ps} = 2.7 \, kW$

$I_{peak} = \dfrac{P_{peak}}{\lambda^2} = 1 \times 10^{16} \, W/m^2$

(Approximate answers are fine for all questions. Relations above are approximately correct for $\Delta\nu$ as FWHM)

Section 6, Question# (3)

General?

Q1. A freshman comes to you and is puzzled by the
following argument regarding the pressure in a U-tube
containing water. His physics professor has told him that
the pressure at points 1 and 2 are the same, and that the
pressure at a point in a fluid is due to the weight of the fluid
above it. The student argues that the pressure at point 1 and
point 2 must be different because they have different
amounts of fluid above them. Provide a simple argument
that explains why the student's argument is wrong.
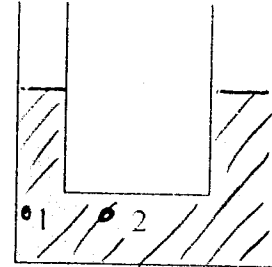
A. One simple way to explain this apparent discrepancy is to imagine that picture
actually represents a vat of liquid with a cup pushed down into it (displacing the
liquid). If we ask what is the force of the water on the cup, it is equal to the
weight of the water displaced. So by the third law, the force of the cup on the
water (i.e. the pressure) is equal to the weight of the fluid displaced. That means
that the pressure on the water at point 2 is the equivalent of the pressure at that
point if the cup were replaced by water, so indeed it is the equivalent of the
weight of the column of water above it!

Problem 1 (Optics):

(M. Shaevitz)

A simple model of an insect eye consists of a spherical array of closely packed conical receptors, each optically isolated from its neighbours. The radius of the spherical array is R. Each cone has a small lens, diameter D, which focuses parallel light onto a photosensitive cell within the cone at a distance r from the center.

a) The light and the angular resolution for each element is limited by the diffraction associated with aperture D. Show that if the insect's eye is simultaneously to achieve maximum sensitivity to light and maximum angular resolving power, the photosensitive cell should lie at $r \approx 2R/3$ from the center.

b) Estimate the diameter of a single lens in the optimised eye and the minimum resolvable angle between two point sources of green light ($\lambda \approx 500$ nm) if $R \approx 1$ mm.
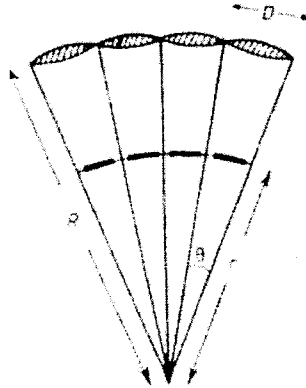
Solution:

a) The focal length of the lens is $R - r$. For maximum sensitivity to light, one wants the diffraction disc to equal the size of the photosensitive cell whose diameter is $d_{disc} \approx rD/R$. The angle of the diffraction disc is $\theta_{min} \approx 1.22 \lambda/D$ giving a diameter at the photosensitive cell of

$$d_{disc} \approx 2.44 \cdot R \cdot \gamma \cdot \lambda/D \approx rD/R$$

For good angular resolution, the fields of view of the cones should be equal to the diffraction limit

$$\theta \approx D/R \approx \theta_{min} \approx 1.22 \lambda/D$$

Solving the two equations gives $r \approx 2R/3$.

b) For $\lambda \approx 500$ nm and $R \approx 1$ mm, $D \approx \sqrt{1.22 \lambda R} \approx 2.5 \cdot 10^{-5} m \approx 25 \mu m$ and $\theta \approx \sqrt{1.22 \lambda/R} \approx 0.025 \approx 1.4°$.