2nd In-Class Test: PHYS7250

April 2009

(20 points)

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1. (GDE, Winter 2008) (8 points)

Problem

A beam of electrons moves with a velocity *v* through a cloud of stationary ions. The beam is cylinddrical with a radius *a* and a uniform electron density. The following parameters are known:

Charge of electron:	*-e*
Mass of electron:	*me*
Number density of electrons:	*ne*
Charge of ion:	*+e*
Mass of ion:	*mi*
Number density of ions:	*ni*

(a)

If *ni = 0.8 ne*, what must be the value of *v* for the beam to be in radial equilibrium (no net radial force on any electron)?

(b)

Does this equilibrium exist in other inertial reference frames? Describe what happens in the reference frame in which electrons are stationary and the ions are moving in the opposite direction. A qualitative explanation is OK, but it must be physically accurate.

Solution

```
$Assumptions = {ni > 0, ne > 0, ϵ0 > 0, μ0 > 0, e > 0, c > 0, v > 0};
```

(a)

To find the force on a electron, we must find the fields.

To find the electric field, use Gauss' Law.

Use cylindrical coordinates. Put the axis down the axis of the cylindrical beam. Construct a Gaussian cylinder of radius *r* and length *L*.

$$\text{eqGaussrlessthana} = \text{Efin}\,2\,\pi\,r\,L == \frac{1}{\epsilon 0}\,(e\,ni - e\,ne)\,\pi\,r^2\,L;$$

```
eqGaussrgreaterthana = Efout 2 π r L == 1/ϵ0 (e ni - e ne) π a^2 L + 1/ϵ0 e ni (π r^2 - π a^2) L;
```

```
Ef[r_, a_] := If[r < a, Efin /. Flatten[Solve[eqGaussrlessthana, Efin]],
   Efout /. Flatten[Solve[eqGaussrgreaterthana, Efout]]] {1, 0, 0}
```

```
Ef[2, 1]
```

$$\left\{\frac{-a^2\, e\, ne + e\, ni\, r^2}{2\, r\, \epsilon 0}, 0, 0\right\}$$

```
Ef[1, 2]
```

$$\left\{-\frac{(e\, ne - e\, ni)\, r}{2\, \epsilon 0}, 0, 0\right\}$$

We see that electric field is inward if *ne* > *ni* so that the force on negatively charged electrons will be in the outward direction.

To find the magnetic field, use Stokes law.

Construct Stokesian circle of radius *r* centered on the axis of the cylinder.

```
eqStokesrlessthana = 2 π r Hfin == -e ne v π r^2;
```

```
eqStokesrgreaterthana = 2 π r Hfout == -e ne v π a^2;
```

where we have put the velocity in the positive *z* direction so the current would be in the negative direction making a magnetic field in the negative φ direction resulting in an inward force on negatively charged electrons to balance the electric force.

```
Hf[r_, a_] := If[r < a, Hfin /. Flatten[Solve[eqStokesrlessthana, Hfin]],
   Hfout /. Flatten[Solve[eqStokesrgreaterthana, Hfout]]] {0, 1, 0}
```

```
Hf[2, 1]
```

$$\left\{0, -\frac{a^2\, e\, ne\, v}{2\, r}, 0\right\}$$

```
Hf[1, 2]
```

$$\left\{0, -\frac{1}{2}\, e\, ne\, r\, v, 0\right\}$$

Note that the way the fields are defined in *Mathematica*, entering values for *r_*, *a_* just tells it which branch to evaluate.

The radial force on an electron is

```
eqForceBalance = 0 == (-e Ef[1, 2] - e v μ0 {0, 0, 1} × Hf[1, 2]).{1, 0, 0}
```

$$0 == \frac{e\, (e\, ne - e\, ni)\, r}{2\, \epsilon 0} - \frac{1}{2}\, e^2\, ne\, r\, v^2\, \mu 0$$

The required *v* is

```
solv = Flatten[Solve[eqForceBalance, v]]
```

$$\left\{v \to -\frac{\sqrt{\frac{ne}{\epsilon 0} - \frac{ni}{\epsilon 0}}}{\sqrt{ne}\, \sqrt{\mu 0}}, v \to \frac{\sqrt{\frac{ne}{\epsilon 0} - \frac{ni}{\epsilon 0}}}{\sqrt{ne}\, \sqrt{\mu 0}}\right\}$$

Since we have made *v* positive,

```
vequil = Simplify[v /. solv[[2]] /. ni → 0.8 ne] /. 1/Sqrt[ϵ0 μ0] → c
```

$0.447214\, c$

in the positive *z* direction.

- **(b)**

To see what happens in another inertial frame, make the transformation with velocity such that the ions are at rest, i.e., a frame moving with the electrons.

```
Λβ = Simplify[Λboost[{0, 0, vequil / c}]]
```

```
Λboost[{0, 0, 0.447214}]
```

```
FemLAB = Fem[Ef[1, 2], μ0 Hf[1, 2]] /. v → vequil /. c^2 -> 1/(ϵ0 μ0)
```

$\text{Fem}\left[\left\{-\frac{(\text{e ne} - \text{e ni})\, r}{2\, \epsilon 0}, 0, 0\right\}, \{0, -0.223607\, \text{c e ne r}\, \mu 0, 0\}\right]$

```
FemNEW = Transpose[Λβ].FemLAB.Λβ;
```

```
Efnew = Simplify[{FemNEW[[1, 2]], FemNEW[[1, 3]], FemNEW[[1, 4]]}] /. ni → 0.8 ne
```

Part::partw : Part 2 of Transpose[Λboost[{0, 0, 0.447214}]] does not exist. ≫

Part::partw : Part 3 of Transpose[Λboost[{0, 0, 0.447214}]] does not exist. ≫

Part::partw : Part 4 of Transpose[Λboost[{0, 0, 0.447214}]] does not exist. ≫

General::stop : Further output of Part::partw will be suppressed during this calculation. ≫

$\Big\{\Big(\text{Transpose}[\Lambda\text{boost}[\{0, 0, 0.447214\}]].\text{Fem}\left[\left\{-\frac{0.1\,\text{e ne r}}{\epsilon 0}, 0, 0\right\}, \{0, -0.223607\,\text{c e ne r}\,\mu 0, 0\}\right].\Lambda\text{boost}[\{0, 0, 0.447214\}]\Big)[[1, 2]],$

$\Big(\text{Transpose}[\Lambda\text{boost}[\{0, 0, 0.447214\}]].\text{Fem}\left[\left\{-\frac{0.1\,\text{e ne r}}{\epsilon 0}, 0, 0\right\}, \{0, -0.223607\,\text{c e ne r}\,\mu 0, 0\}\right].\Lambda\text{boost}[\{0, 0, 0.447214\}]\Big)[[1, 3]],$

$\Big(\text{Transpose}[\Lambda\text{boost}[\{0, 0, 0.447214\}]].\text{Fem}\left[\left\{-\frac{0.1\,\text{e ne r}}{\epsilon 0}, 0, 0\right\}, \{0, -0.223607\,\text{c e ne r}\,\mu 0, 0\}\right].\Lambda\text{boost}[\{0, 0, 0.447214\}]\Big)[[1, 4]]\Big\}$

This is what we expect, i.e., the equilibrium holds in this frame because the particle densities change due to the length contraction.

This is difficult to demonstrate in general, but phyiscally, we certainly expect equilibrium to be an invariant observation.

2. (GDE, Fall 2001) (8 points)

Problem

A charged particle in an accelerator is moving in a circular orbit of radius *r* with a relativistic speed *v*, where $\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} >> 1$

(a)

Why is the power radiated by the charge Lorentz invariant?

(b)

Calculate the total power radiated by the charge when you are given an invariant form of the radiated power, i.e.,

$$P = \frac{-q^2 \sqrt{\frac{\mu_0}{\epsilon_0}}}{6 \pi m^2 c^2} \frac{dp_\alpha}{d\tau} \frac{dp^\alpha}{d\tau}.$$

where the 4-vector summation convention is used.

(c)

You want to observe the radiated power. Where do you set up your antenna relative to the particle orbit?

(d)

What is the peak frequency of the observed power? Does it have any relation to the period of the particle?

Solution

(a)

Power is energy over time. Both are the 0th components of a 4-vector, so we would expect them to transform in a similar fashion so that their ratio stays the same.

(b)

In 4-vector notation, Newton's Laws give us directly the power.

The force 3-vector is in Cartesian coordinates with the circular orbit in the *x*-*y* plane

$$\texttt{force3V} = \left\{\gamma\, m\, \gamma \frac{v^2}{r} \mathrm{Cos}[\varphi],\ \gamma\, m\, \gamma \frac{v^2}{r} \mathrm{Sin}[\varphi],\ 0\right\};$$

The first factor of γ comes from the fact the we need γ *Force* and the second comes from the relativistic increase in the mass.

The force 4-vector is

```
force4V = {0, force3V[[1]], force3V[[2]], force3V[[3]]};
```

$$\text{power} = \text{Simplify}\left[\frac{-q^2 \sqrt{\frac{\mu_o}{\epsilon_o}}}{6\,\pi\, m^2\, c^2}\ \text{force4V}.\{0, -\text{force3V}[[1]], -\text{force3V}[[2]], -\text{force3V}[[3]]\}\right]$$

$$\frac{q^2\, v^4\, \gamma^4 \sqrt{\frac{\mu_o}{\epsilon_o}}}{6\, c^2\, \pi\, r^2}$$

This is the SI form of Eq. (14.31) in Jackson.

- **(c)**

We know the power is forward peaked. So we would set up our antenna along a tangent to the circle. But that could be anywhere.

We certainly want to be in the plane of the orbit, however.

- **(d)**

We discussed in class that the peak frequency is much higher than the orbital frequency. The emission is a broad spectrum with a critical frequency of order γ^3times the orbital frequency.

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3. (GDE, Winter 1996) (4 points)

- ### Problem

In the process of Compton scattering, an x-ray photon of energy E_0 scatters from an electron at rest.

- **(a)**

Find the energy E_{sc} of the scattered photon as a function of the scattering angle θ.

- **(b)**

Express your answer in the "traditional" form of the scattered photon wavelength λ_{sc}as a function of the incident photon wavelength λ_0.

- ### Solution

```
$Assumptions = {Esc > 0, E0 > 0, m > 0, c > 0, θ ∈ Reals, φ ∈ Reals};
```

- **(a)**

We will use conservation of energy and momentum.

$$\text{eqEconservation} = E0 + m\,c^2 == Esc + \sqrt{(pe\,c)^2 + \left(m\,c^2\right)^2};$$

```
eqPxconservation = E0/c == Esc/c Cos[θ] + pe Cos[φ];
eqPyconservation = 0 == Esc/c Sin[θ] - pe Sin[φ];
```

We could use energy conservation to eliminate *pe* and then conservation of momentum in the *y* direction to eliminate φ, the exit angle of the electron, and then finally find *Esc* as a function of *E0* and θ as desired.

Instead

```
sola = Flatten[Solve[{eqEconservation, eqPxconservation, eqPyconservation}, {Esc, pe, φ}]]
```

```
$Aborted
```

Mathematica seems to be having trouble.

```
solpe = Flatten[Solve[eqEconservation, pe]]
```

$$\left\{pe \to -\frac{\sqrt{E0^2 - 2\,E0\,Esc + Esc^2 + 2\,c^2\,E0\,m - 2\,c^2\,Esc\,m}}{c},\ pe \to \frac{\sqrt{E0^2 - 2\,E0\,Esc + Esc^2 + 2\,c^2\,E0\,m - 2\,c^2\,Esc\,m}}{c}\right\}$$

```
eq1a = eqPxconservation /. solpe[[2]]
```

$$\frac{E0}{c} == \frac{Esc\,Cos[\theta]}{c} + \frac{\sqrt{E0^2 - 2\,E0\,Esc + Esc^2 + 2\,c^2\,E0\,m - 2\,c^2\,Esc\,m}\ Cos[\varphi]}{c}$$

```
eq2a = eqPyconservation /. solpe[[2]]
```

$$0 == \frac{Esc\,Sin[\theta]}{c} - \frac{\sqrt{E0^2 - 2\,E0\,Esc + Esc^2 + 2\,c^2\,E0\,m - 2\,c^2\,Esc\,m}\ Sin[\varphi]}{c}$$

```
solφ = Flatten[Solve[eq2a, Sin[φ]]]
```

$$\left\{Sin[\varphi] \to \frac{Esc\,Sin[\theta]}{\sqrt{(-E0 + Esc)\,\left(-E0 + Esc - 2\,c^2\,m\right)}}\right\}$$

```
eq1b = Simplify[eq1a /. Cos[φ] → √(1 - Sin[φ]^2) /. solφ]
```

$$Esc\,Cos[\theta] + \sqrt{(E0 - Esc)\,\left(E0 - Esc + 2\,c^2\,m\right)}\ \sqrt{1 - \frac{Esc^2\,Sin[\theta]^2}{(E0 - Esc)\,\left(E0 - Esc + 2\,c^2\,m\right)}} == E0$$

```
solEsc = Flatten[Solve[eq1b, Esc]]
```

$$\left\{Esc \to \left(-2\,E0 - 2\,c^2\,m + 2\,E0\,Cos[\theta] - \sqrt{\left(2\,E0 + 2\,c^2\,m - 2\,E0\,Cos[\theta]\right)^2 + 8\,c^2\,E0\,m\,\left(-1 + Cos[\theta]^2 + Sin[\theta]^2\right)}\right) \Big/ \left(2\,\left(-1 + Cos[\theta]^2 + Sin[\theta]^2\right)\right),\ Esc \to \left(-2\,E0 - 2\,c^2\,m + 2\,E0\,Cos[\theta] + \sqrt{\left(2\,E0 + 2\,c^2\,m - 2\,E0\,Cos[\theta]\right)^2 + 8\,c^2\,E0\,m\,\left(-1 + Cos[\theta]^2 + Sin[\theta]^2\right)}\right) \Big/ \left(2\,\left(-1 + Cos[\theta]^2 + Sin[\theta]^2\right)\right)\right\}$$

Mathematica gets a bad solution for this too.

eq1c =

$$\text{Simplify}\left[\left(\sqrt{(\text{E0}-\text{Esc})\,(\text{E0}-\text{Esc}+2\,c^2\,m)}\,\sqrt{1-\frac{\text{Esc}^2\,\text{Sin}[\theta]^2}{(\text{E0}-\text{Esc})\,(\text{E0}-\text{Esc}+2\,c^2\,m)}}\right)^2 == (\text{E0}-\text{Esc}\,\text{Cos}[\theta])^2\right]$$

$$\text{E0}\,(c^2\,m+\text{Esc}\,\text{Cos}[\theta]) == \text{Esc}\,(\text{E0}+c^2\,m)$$

solEsc = Flatten[Solve[eq1c, Esc]]

$$\left\{\text{Esc}\to\frac{c^2\,\text{E0}\,m}{\text{E0}+c^2\,m-\text{E0}\,\text{Cos}[\theta]}\right\}$$

This looks reasonable.

- **(b)**

We can relate energy of the photon to the wavelength in the following manner.

$$\text{eq}\lambda = \text{energy} == \text{hbar}\,\omega\ /.\ \omega\to k\,c\ /.\ k\to\frac{2\,\pi}{\lambda}$$

$$\text{energy} == \frac{2\,c\,\text{hbar}\,\pi}{\lambda}$$

So

$$\text{eqCompton} = \frac{2\,c\,\text{hbar}\,\pi}{\lambda\text{sc}} == \frac{c^2\,\text{E0}\,m}{\text{E0}+c^2\,m-\text{E0}\,\text{Cos}[\theta]}\ /.\ \text{E0}\to\frac{2\,c\,\text{hbar}\,\pi}{\lambda 0}$$

$$\frac{2\,c\,\text{hbar}\,\pi}{\lambda\text{sc}} == \frac{2\,c^3\,\text{hbar}\,m\,\pi}{\lambda 0\left(c^2\,m+\frac{2\,c\,\text{hbar}\,\pi}{\lambda 0}-\frac{2\,c\,\text{hbar}\,\pi\,\text{Cos}[\theta]}{\lambda 0}\right)}$$

solλ = Simplify[Flatten[Solve[eqCompton, λsc]]]

$$\left\{\lambda\text{sc}\to\frac{2\,\text{hbar}\,\pi+c\,m\,\lambda 0-2\,\text{hbar}\,\pi\,\text{Cos}[\theta]}{c\,m}\right\}$$

i.e.,

$$\lambda\text{sc} = \lambda 0+\frac{2\,\pi\,\text{hbar}\,c}{m\,c^2}\,(1-\text{Cos}[\theta])\ /.\ \frac{2\,\pi\,\text{hbar}\,c}{m\,c^2}\to\lambda\text{Compton}$$

$$\lambda 0+\lambda\text{Compton}\,(1-\text{Cos}[\theta])$$

where *λCompton* of the electron is 2.4263 10^{-12} *m*.