Oscillatory Motion

In the beginning of the course, we studied the position function and some cases of one-dimensional motion. There were three cases: x = const (constant position function); v = const (linear position function); a = const (quadratic position function). The last one (accelerated motion) gave us the two kinematic equations which we used in solving many interesting problems.

Oscillatory motion is yet another case of one-dimensional motion that sometimes takes place. Imagine, for example, a ball of mass m attached to a spring of stiffness k and constrained to move on the horizontal table in one dimension.

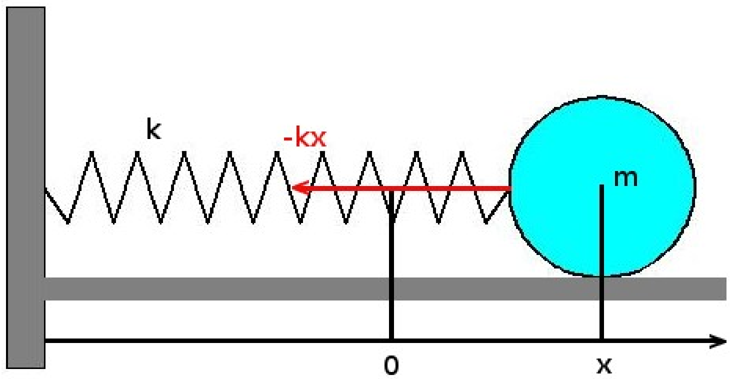


Fig. 1

Let us place the origin at the point where the ball is in equilibrium, then the spring's displacement will equal the position of the ball. Setting up second Newton's law on the horizontal direction, we have

$$m a = -k x \quad ,$$

since the only horizontal force on the ball is the force from the spring, which, by Hooke's law, equals kx and it has a negative projection on our x-axis. Now the goal is to solve for the motion of the ball (the ball's position function $x(t)$).

$m a = -k x$ is the only equation we can get for this system. Remembering that $a = \frac{d^2 x}{dt^2}$, we can rewrite it as

$$m \frac{d^2 x}{dt^2} = -k x \quad .$$

It is a second-order differential equation with respect to x. One way to solve it is first switch to the variable of speed:

$m \frac{dv}{dt} = -k x$ (our Newton's equation with speed instead of acceleration or position);

$\frac{dx}{dt} = v$ (definition of speed: first time-derivative of position).

Dividing them by each other, we have $m\frac{dv}{dx}=-k\frac{x}{v}$, or $m\,v\,dv=-k\,x\,dx$. Since the variables separated (v's on one side and x's on the other), this can be integrated into $\frac{mv^2}{2}=-\frac{kx^2}{2}+C$, or $\frac{mv^2}{2}+\frac{kx^2}{2}=C$, where C is the constant of integration. In this formula, of course, you recognize the energy conservation equation (see how we derived it in the Energy chapter: we integrated the second Newton's law), so the constant of integration is just the total energy E: $\frac{mv^2}{2}+\frac{kx^2}{2}=E$.

At the highest position $x = A$ the ball will stop: all kinetic energy will be transferred into the potential energy of the spring. Therefore, the total energy equals the potential energy of the spring at $x = A$: $E=\frac{kA^2}{2}$. As the spring will be pulling back, the ball will go to the left and achieve the minimum position $x = -A$ (due to conservation of energy: when at rest, the potential energy must be $\frac{kA^2}{2}$, so the turning points must be A and $-A$) where it will stop again and resume moving to the right. Thus, we can plug in the value of E in our conservation equation:

$\frac{mv^2}{2}+\frac{kx^2}{2}=\frac{kA^2}{2}$, or $mv^2=k(A^2-x^2)$, from where we have $v=\sqrt{\frac{k}{m}(A^2-x^2)}$.

Now, remembering that $v=\frac{dx}{dt}$ by definition, we plug it in and have $\frac{dx}{dt}=\sqrt{\frac{k}{m}(A^2-x^2)}$. Separating the variables again to be able to integrate (x's on one side and t's on the other), we have $dt=\frac{dx}{\sqrt{\frac{k}{m}(A^2-x^2)}}=\sqrt{\frac{m}{k}}\frac{dx}{\sqrt{A^2-x^2}}$. We will need to integrate this to get $t(x)$ and then reverse the function to finally have $x(t)$.

So, $t=\sqrt{\frac{m}{k}}\int\frac{dx}{\sqrt{A^2-x^2}}=\sqrt{\frac{m}{k}}\arcsin\frac{x}{A}+C_1$, where C_1 is another constant of integration. Pulling out x, we have $t-C_1=\sqrt{\frac{m}{k}}\arcsin\frac{x}{A}$, $\arcsin\frac{x}{A}=\sqrt{\frac{k}{m}}t-\sqrt{\frac{k}{m}}C_1$, $x=A\sin(\sqrt{\frac{k}{m}}t-\sqrt{\frac{k}{m}}C_1)$.

We finally have the position function of the ball. It is a sine function, periodic, with the maximum A and minimum $-A$, as expected. The coefficient at the time variable, $\sqrt{\frac{k}{m}}$, determines how fast it oscillates, so it is called the *frequency*: $\omega=\sqrt{\frac{k}{m}}$. The maximum position A is called the *amplitude* (that's why we called it A in the first place). Finally, the constant $-\sqrt{\frac{k}{m}}C_1$ is a (scaled) constant of integration and determines the initial position of the ball ($x(0)=A\sin(-\sqrt{\frac{k}{m}}C_1)$), so it is called the *phase*: $\varphi = -\sqrt{\frac{k}{m}} C_1$. Thus, the ball's position function can be rewritten in terms of the quantities defined:

$$x = A\sin(\omega t + \varphi) \ .$$

This is the generic position function of all kinds of similar oscillatory motion. The motion with this position function is called *harmonic*. (Motion can be oscillatory but not harmonic, e.g., a ball elastically bouncing up and down on the horizontal surface.)

Kinematics of harmonic motion. We shall now study some properties of this type of motion. First, let the sine function not confuse you: there are no angles here whose sine is being taken. Sine is solely a function and we have the position versus time dependence in the fashion of a sine function, that's all. However, there is some correspondence to uniform circular motion here.

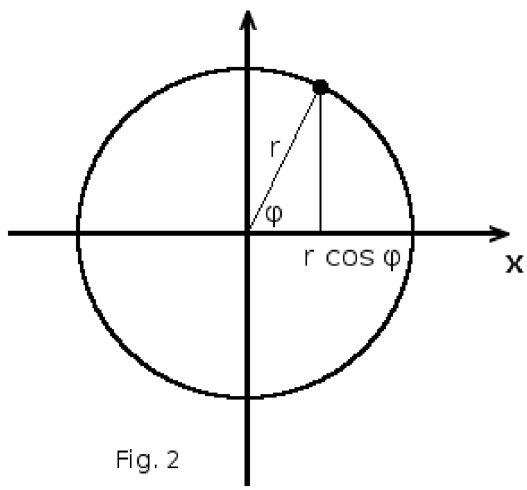


Fig. 2

Imagine a point rotating around a circle with constant speed. Its *x*-coordinate at all times will be *r* cos φ, where φ is the angle the particle's radius-vector is currently making with the *x*-axis. But from rotational kinematics we know that φ can be viewed as the angular position, and for uniform circular motion with constant *v* (and therefore constant ω), its angular position function is $\varphi = \varphi_0 + \omega t$. So the *x*-coordinate of the point changes with time as $x(t) = r\cos\varphi = r\cos(\omega t + \varphi_0)$, which is exactly the harmonic motion! Let the cosine instead of sine not confuse you: they are essentially the same function, only shifted by *π*/2 against each other: $\sin(x + \frac{\pi}{2}) = \sin x \cos\frac{\pi}{2} + \cos x \sin\frac{\pi}{2} = 0 \cdot \sin x + 1 \cdot \cos x = \cos x$. Thus, any function $A\sin(\omega t + \varphi)$ cane be transformed into $A\cos(\omega t + \psi)$, where ψ is another phase constant. Let's make sure:

$$A\sin(\omega t + \varphi) = A\sin((\omega t + \varphi - \frac{\pi}{2}) + \frac{\pi}{2}) = A\cos(\omega t + \varphi - \frac{\pi}{2}) = A\cos(\omega t + \psi)$$, so any sine harmonic function equals the cosine function with the new phase $\psi = \varphi - \frac{\pi}{2}$.

There are four cases where it is possible to write out the position function without the phase. Let's start with the easiest one: $x = A\sin\omega t$. It equals zero at time zero, so here it means that you started your clock when the particle was at its initial position. Further, it is positive at small positive *t*, so the particle was moving to the right (it entered the positive half of the axis at time zero).

When $\varphi=\frac{\pi}{2}$, $x=A\sin(\omega t+\varphi)=A\sin(\omega t+\frac{\pi}{2})=A\cos\omega t$. Here at zero time $x = A$. That means you start your clock when the particle is at the rightmost position.

When $\varphi=\pi$, $x=A\sin(\omega t+\pi)=A(\sin\omega t\cos\pi+\cos\omega t\sin\pi)=-A\sin\omega t$. The particle is again at zero at time zero, but its position grows negative with positive time: the particle is passing the equilibrium and moving to the left, entering the negative half-axis.

When $\varphi=\frac{3\pi}{2}$, $x=A\sin(\omega t+\frac{3\pi}{2})=A(\sin\omega t\cos\frac{3\pi}{2}+\cos\omega t\sin\frac{3\pi}{2})=-A\cos\omega t$. Here $x(0)=-A\cos 0=-A$. The particle is at the leftmost position $x = -A$.

The particle's speed and acceleration can be obtained by differentiating the position function.

$$v(t)=\frac{d}{dt}A\sin(\omega t+\varphi)=\omega A\cos(\omega t+\varphi) \quad ,$$
$$a(t)=\frac{d}{dt}\omega A\cos(\omega t+\varphi)=-\omega^2 A\sin(\omega t+\varphi) \quad .$$

Notice that $a(t)=-\omega^2 x(t)$, and, since $\omega^2=\frac{k}{m}$, $a(t)=-\frac{k}{m}x(t)$, or $ma(t)=-kx(t)$, which is our original Newton's equation. Thus, in harmonic motion, the acceleration (second derivative of the position) is proportional to the position with the minus sign, and the coefficient of proportionality is the frequency squared.

From the equations above it follows that the maximum values of speed and acceleration are $v_{max}=\omega A$, $a_{max}=\omega^2 A$ (sine and cosine cannot be greater than 1 or less than –1). If you look at the phases, you will see that speed has its maximum magnitude at $x = 0$ (which is intuitive because the spring potential is zero there and all energy is transferred to kinetic), and the acceleration has its maximum magnitude at the turning points $\pm A$ (it is also easy to understand because the net force on the particle from the spring is the greatest at the farthest points).

Problem 1. A particle is undergoing one-dimensional harmonic motion. At the position x_1 its speed was measured to be v_1, and at the position x_2 it was v_2. Find the amplitude and the frequency of motion.

Solution. The generic position function of the harmonic motion is

$$x=A\sin(\omega t+\varphi) \quad .$$

The speed of the particle at any time is

$$v=\omega A\cos(\omega t+\varphi) \quad .$$

Let's call the moments of time when the two measurements were made t_1 and t_2 respectively, and write out the given positions and speeds:

$$\begin{cases} x_1 = A\sin(\omega t_1+\varphi) \\ v_1 = \omega A\cos(\omega t_1+\varphi) \\ x_2 = A\sin(\omega t_2+\varphi) \\ v_2 = \omega A\cos(\omega t_2+\varphi) \end{cases}$$

We need to solve for ω and *A*. Let's rewrite the system like this:

$$\begin{cases} \frac{x_1}{A} = \sin(\omega t_1+\varphi) \\ \frac{v_1}{\omega A} = \cos(\omega t_1+\varphi) \\ \frac{x_2}{A} = \sin(\omega t_2+\varphi) \\ \frac{v_2}{\omega A} = \cos(\omega t_2+\varphi) \end{cases}$$

and then square and add ## 1 and 2 and ## 3 and 4 (when you have sine and cosine with unneeded variables inside, it's convenient to single them out, square and add, so that they both disappear since $\sin^2 x+\cos^2 x=1$):

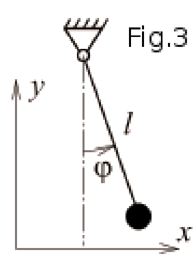
$$\begin{cases} \frac{x_1^2}{A^2}+\frac{v_1^2}{\omega^2 A^2}=1 \\ \frac{x_2^2}{A^2}+\frac{v_2^2}{\omega^2 A^2}=1 \end{cases}, \text{ or } \begin{cases} x_1^2+\frac{v_1^2}{\omega^2}=A^2 \\ x_2^2+\frac{v_2^2}{\omega^2}=A^2 \end{cases}.$$

Now this one has only *ω* and *A* as unknowns, so it's easy to solve for them:

$$\omega=\sqrt{\frac{v_2^2-v_1^2}{x_1^2-x_2^2}}\ , \quad A=\sqrt{\frac{v_1^2 x_2^2-v_2^2 x_1^2}{v_1^2-v_2^2}}\ .$$

Applications of harmonic motion theory.

Mathematical pendulum. A material point (object of a negligible size), for example, a small ball, suspended on an ideal (massless and non-stretchable) string is called a mathematical pendulum (see Fig. 3 below).



Given the mass *m* of the ball and length ℓ of the string, let's find the frequency of its oscillations.

The ball moves along a circular arc of radius ℓ and, when the string makes angle φ with the vertical, is acted upon by the force of gravity *mg*, which has the projection *mg* sin φ along the tangent line to the circle, and by the tension of the string *T* which has no tangential projection. The ball experiences normal acceleration towards the pivot, but we are not interested in it right now; what we want to pay attention to is its *tangential* acceleration, perpendicular to the radius. The tangential projection of the second Newton's law will give us:

$$m a_\tau = m g \sin\varphi \quad .$$

Further, we can connect the tangential acceleration with the angle φ, knowing that the angular acceleration $\varepsilon = \frac{d^2\varphi}{dt^2}$ and the tangential acceleration a_τ are proportional:

$$\varepsilon = -\frac{a_\tau}{\ell} \quad ,$$

the minus sign coming from the fact that when the angle diminishes, acceleration grows.

Plugging it all in, we have

$$m\varepsilon\ell = -m g \sin\varphi \quad , \quad \varepsilon = \frac{d^2\varphi}{dt^2} = -\frac{g}{\ell}\sin\varphi \quad .$$

The equation $\frac{d^2\varphi}{dt^2} = -\frac{g}{\ell}\sin\varphi$ is a second-order differential equation with respect to φ. Moreover, given as it is, it's impossible to solve it in elementary functions. We need to make approximations to move on. The first-order approximation could be that the angle φ is much less than π/2 or 1, so $\sin\varphi \approx \varphi$, and our equation becomes $\frac{d^2\varphi}{dt^2} = -\frac{g}{\ell}\varphi$.

Let's look above. We have worked on this equation before! In math, it doesn't matter whether you call the variable *x* or φ: the same equation will yield the same solution. And what we have here is the harmonic motion equation with respect to φ: the second time-derivative of it is proportional to it with the minus sign. And the coefficient of proportionality, which here is $\frac{g}{\ell}$, must be the frequency squared. Thus, $\omega = \sqrt{\frac{g}{\ell}}$.

The period of motion is the time during which the motion completes one oscillation. Looking at the generic position function $x = A\sin(\omega t + \varphi)$ (do not confuse the phase φ with the angular position φ for the pendulum above!), we see that the period $T = \frac{2\pi}{\omega}$. There is the third parameter, called the *linear frequency* (therefore, ω is sometimes referred to as the angular frequency) $\nu = \frac{1}{T} = \frac{\omega}{2\pi}$, denoted by the Greek letter nu (ν) (in older books it is often denoted by the Latin letter *f*); it shows the number of oscillations per second and measured in Hz. Angular frequency, like angular speed, is measured in s^{-1}.

Therefore, the period of one small oscillation of the mathematical pendulum is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}}$, and for the ball on the spring $T = 2\pi\sqrt{\frac{m}{k}}$. The period of the mathematical pendulum, as it turns out, does not depend on its mass or amplitude, only on its length! But remember that we solved the pendulum only for small oscillations; when the angular amplitude is comparable to π/2, the oscillation will not be harmonic and the period will depend on the amplitude. Likewise, for the ball on the spring we used Hooke's law, which works well for small deformations of the spring; for large amplitudes, the spring will be affected differently and the oscillations will stop being harmonic.

I made a plot which shows how the period of oscillations of the mathematical pendulum is affected by its amplitude.

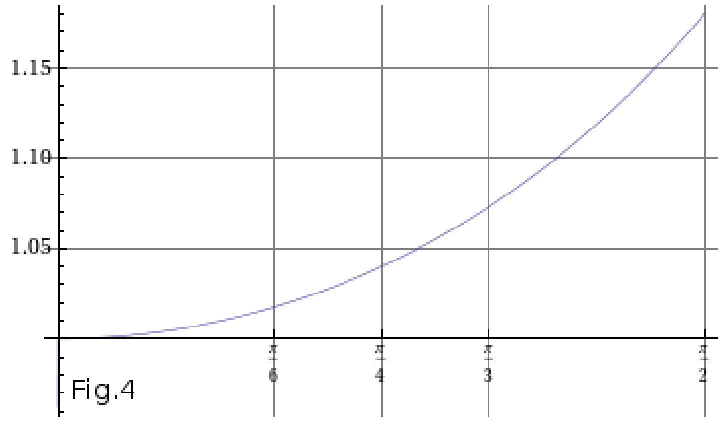


Fig.4

On the horizontal you have the angular amplitude while on the vertical you have the factor by which the period differs from the one of small oscillations $T_0 = 2\pi\sqrt{\frac{\ell}{g}}$. You can see that even for π/6 (30º) the increase is still insignificant: T = 1.02 T_0. For larger amplitudes, the error that our harmonic approximation gives will be more noticeable: for π/3, the error would already be 7% (T = 1.07 T_0).

Physical pendulum. All real pendula are physical. In general, a physical pendulum is a rigid body pivoted around some point and left to oscillate due to gravity force. We shall here find the period of oscillation of a physical pendulum.

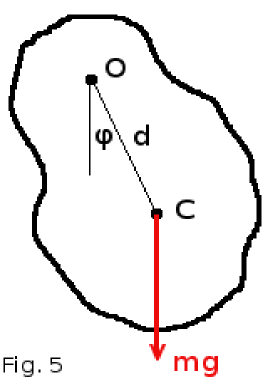


Fig. 5

Consider a random rigid body pivoted around the axis going through point O normal to the paper, whose center of mass is at point C. It is at equilibrium when the line OC is vertical (the torque of the gravity force would be zero). When the body is inclined by angle φ, the torque of gravity would be $mgd\sin\varphi$, where d is the distance between the pivot and the center of mass. Setting up the torque equation:

$I\varepsilon = mgd\sin\varphi$, where I is the rigid body's moment of inertia around the point O.

Since $\varepsilon = -\frac{d^2\varphi}{dt}$ (angular acceleration is positive when the angle diminishes),

$$I\frac{d^2\varphi}{dt} = -mgd\sin\varphi \text{ , or } \frac{d^2\varphi}{dt} = -\frac{mgd}{I}\sin\varphi \text{ .}$$

We have the same type of equation as for the mathematical pendulum. Approximating $\sin\varphi \approx \varphi$ for small oscillations, we have

$$\frac{d^2\varphi}{dt} = -\frac{mgd}{I}\varphi \text{ .}$$

This is the harmonic motion equation again. From our experience, we know that $\frac{mgd}{I}$ must be the frequency squared:

$$\omega^2 = \frac{mgd}{I} \text{ , } \omega = \sqrt{\frac{mgd}{I}} \text{ , } T = 2\pi\sqrt{\frac{I}{mgd}} \text{ .}$$

Problem 2. Find the period of small oscillations of a) a rod of length ℓ about its end; b) a hoop of radius R about any point on it.

Solution. Using $T = 2\pi\sqrt{\frac{I}{mgd}}$, we have:

a) For the rod about its end $I = \frac{m\ell^2}{3}$ and distance from the center of mass to the pivot is $d = \frac{\ell}{2}$, so

$$T = 2\pi\sqrt{\frac{m\ell^2 \cdot 2}{3 \cdot mg\ell}} = 2\pi\sqrt{\frac{2\ell}{3g}} \text{ .}$$

b) The hoop's moment of inertia about its center is mR^2. Using Huygens – Steiner theorem, we have $I = mR^2 + mR^2 = 2mR^2$ about any point on the hoop. Obviously, $d = R$ here. Plugging in:

$$T = 2\pi\sqrt{\frac{2mR^2}{mgR}} = 2\pi\sqrt{\frac{2R}{g}} \text{ .}$$

Problem 3. Remembering that in the real world the mathematical pendulum is actually a solid ball of radius r suspended by the string of length ℓ, find its period of small oscillations taking this fact into account. Compare it to the formula derived for the mathematical pendulum.

Solution. Using the formula for the period of oscillations of a physical pendulum

$$T=2\pi\sqrt{\frac{I}{mgd}}$$

and seeing that for this situation the distance between the pivot and the center of mass would be $d=\ell+r$, and the moment of inertia of the solid ball around the pivot is $I=\frac{2}{5}mr^2+m(\ell+r)^2$, we have

$$T=2\pi\sqrt{\frac{\frac{2}{5}mr^2+m(\ell+r)^2}{mg(\ell+r)}}=2\pi\sqrt{\frac{\ell+r}{g}+\frac{2r^2}{5g(\ell+r)}} \quad .$$

If $r\to 0$, our formula becomes the standard period of the mathematical pendulum $T=2\pi\sqrt{\frac{\ell}{g}}$.

Let's try to consider the situation when $r \ll \ell$, but still not zero. For this case,

$T\approx 2\pi\sqrt{\frac{\ell}{g}+\frac{r}{g}+\frac{2r^2}{5g\ell}}=2\pi\sqrt{\frac{\ell}{g}}\sqrt{1+\frac{r}{\ell}+\frac{2r^2}{5\ell^2}}\approx 2\pi\sqrt{\frac{\ell}{g}}\sqrt{1+\frac{r}{\ell}}\approx 2\pi\sqrt{\frac{\ell}{g}}\left(1+\frac{r}{2\ell}\right)$, thus the error arising for the mathematical pendulum model's actually being physical is proportional to half the ratio of the ball radius and the strings length. Thus, a small ball on a long string would suffice for a good model of the mathematical pendulum.

More complicated oscillators. Sometimes you may encounter an oscillating system that is more than a pendulum: for example, a rod with attached springs and strings, some of which go over some pulleys, etc. The usual question is to find the frequency (or period, or linear frequency) of the oscillator. There are two ways to do that. First is the torque equation: in a non-equilibrium position find the torques and set them equal to $I\varepsilon$. Many of them will depend on the angle φ, and ε would be the negative second time-derivative of φ. Now make approximations for small φ ($\sin\varphi\approx\varphi, \cos\varphi\approx 1$), and pull ε on one side. Your equation would look like this:

$-\varepsilon=\frac{d^2\varphi}{dt^2}=-(\ldots)\varphi$; this is the harmonic equation, and the expression in the parentheses must be the frequency squared. You have found the frequency.

The second way is to write out the potential and kinetic energy of the oscillator in a non-equilibrium position and set it equal to the constant E. Then differentiate the whole expression by time. Angular speed should cancel and you will be left essentially with the torque equation. Now proceed as above and find the frequency.

Problem 4. A rod of mass m and length ℓ is pivoted around one of its ends. At the length $\ell/3$ from the pivot, a ball of the same mass m is attached to the rod, and at the other end of the rod a spring of stiffness k is attached, whose other end is fixed. Find the frequency of this oscillator.

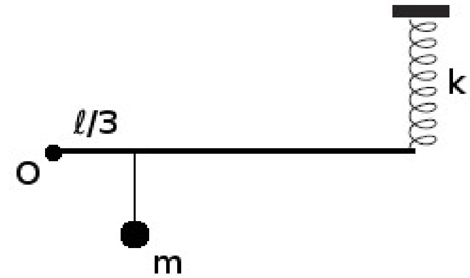


Fig. 6

Solution (torques). When you rise the rod by some angle φ, the torque from the hanging ball will be $mg\ell/3\cos\varphi$, the torque from the gravity on the rod will be $mg\ell/2\cos\varphi$, and from the spring $k(x-x_0)\ell$, where x is its displacement from the horizontal up, and x_0 its original displacement from the non-stretched position. The moment of inertia of the rod and ball is

$$I=\frac{m\ell^2}{3}+m\left(\frac{\ell}{3}\right)^2=\frac{4}{9}m\ell^2 \quad .$$

From geometry, $x=\ell\sin\varphi\approx\ell\varphi$.

The torque equation is

$$\frac{4}{9}m\ell^2\varepsilon=\frac{mg\ell}{3}\cos\varphi+\frac{mg\ell}{2}\cos\varphi+k(\ell\varphi-x_0)\ell\approx\frac{mg\ell}{3}+\frac{mg\ell}{2}+k(\ell\varphi-x_0)\ell=\frac{5}{6}mg\ell+k(\ell\varphi-x_0)\ell \quad .$$

$$\frac{4}{9}\varepsilon=\frac{5g}{6\ell}+\frac{k}{m}\left(\varphi-\frac{x_0}{\ell}\right) \quad .$$

The additional constants in the harmonic equation don't affect the frequency: if, instead of $\ddot{x}=-\omega^2 x$ we have $\ddot{x}=-\omega^2 x+C$, where C is any constant (dot above the variable is a time-derivative, two dots – second time-derivative), it affects only the equilibrium position, but not frequency (you can try proving it as an exercise). So, our torque equation,

$$\varepsilon=\frac{9}{4}\frac{5}{6}\frac{g}{\ell}+\frac{9}{4}\frac{k}{m}\varphi-\frac{k}{m}\frac{x_0}{\ell} \quad ,$$

will give us the frequency $\omega=\sqrt{\frac{9k}{4m}}$.

Solution (energy). Displaced by φ, the setup has the potential energy of the ball $\frac{mg\ell}{3}\sin\varphi$, potential energy of the rod $\frac{mg\ell}{2}\sin\varphi$ and potential energy of the spring $k\frac{(x-x_0)^2}{2}=k\frac{(\ell\varphi-x_0)^2}{2}$. The kinetic energy of the system is $\frac{I\omega^2}{2}=\frac{1}{2}\frac{4}{9}m\ell^2\omega^2=\frac{2}{9}m\ell^2\omega^2$. Conservation of energy:

$$E=\frac{2}{9}m\ell^2\dot{\varphi}^2+\frac{mg\ell}{3}\sin\varphi+\frac{mg\ell}{2}\sin\varphi+k\frac{(\ell\varphi-x_0)^2}{2}$$, or

$$E=\frac{2}{9}m\ell^2\dot{\varphi}^2+\frac{5}{6}mg\ell\sin\varphi+k\frac{(\ell\varphi-x_0)^2}{2}\approx\frac{2}{9}m\ell^2\dot{\varphi}^2+\frac{5}{6}mg\ell\varphi+k\frac{(\ell\varphi-x_0)^2}{2} \quad .$$

Differentiating the whole thing by time:

$$0=\frac{2}{9}m\ell^2 2\dot{\varphi}\ddot{\varphi}+\frac{5}{6}mg\ell\dot{\varphi}+\frac{k}{2}2(\ell\varphi-x_0)\ell\dot{\varphi} \quad , \quad \dot{\varphi}$$ cancels, divide by $m\ell^2$, and we have

$$0=\frac{4}{9}\ddot{\varphi}+\frac{5}{6}\frac{g}{\ell}+\frac{k}{m}\varphi-\frac{k}{m}\frac{x_0}{\ell} \quad , \quad \ddot{\varphi}=-\frac{9}{4}\frac{k}{m}\varphi+\left(\frac{kx_0}{m\ell}-\frac{5g}{6\ell}\right) \quad .$$

The quantity in the parentheses is a constant and it does not affect the solution, so the coefficient before φ must be frequency squared:

$$\omega^2=\frac{9k}{4m} \quad , \quad \omega=\sqrt{\frac{9k}{4m}} \quad .$$

In many cases, deriving energy is easier than setting up the torque equation.