Rotational Kinematics

In many situations, motion occurs in a circle. How many coordinates do we need to fully describe such motion? Knowing that a circle is a two-dimensional object, one might say that we need two coordinates, say, *x* and *y*. But the geometry of a circle already binds these two coordinates into a relation $(x - x_0)^2 + (y - y_0)^2 = R^2$ (*R* being the radius of the circle and (x_0, y_0) the coordinates of its center) for any point on it, so one of them can always be expressed through the other. Thus, we need just one coordinate to define the point's position on a circle.

It can be, for example, the length of the circular path from a certain starting point, but a more conventional measure would be the angle with the radius directed at our point and the radius directed at some chosen starting point in the counterclockwise direction, as in Fig 1.

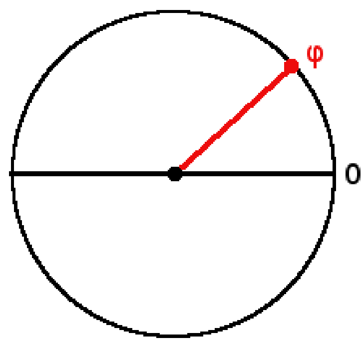

Fig. 1

Our point, shown in red in Fig. 1, has the *angular position* φ. From here we can draw an analogy with the linear position we had in the beginning of the course: there we had an infinite line and the point's position was determined by the place of the origin and the scale factor (the length which is defined to be one unit), and here it is determined solely by the angle of the point's radius with the "zero radius" or the "origin" on the circle.

But wait. Which angle should we consider when the particle is slightly below the zero point – is it a small negative angle or the big positive one (almost 360˚)? Likewise, the point diametrically opposite to the starting point equally has a right to take the position of 180˚ and -180˚. We have to work out some precision in our definitions. Further, if the point moves from its initial position φ one full circle (and obviously comes to the same place it was before), does it take the same position φ or φ+360˚? If it is the same position, how then would we know if it moved one full circle, 26 full circles, or didn't move at all?

Of course, such obstacles arise from the fact that the circle has finite length. Our x-axis in linear kinematics was infinite; if a particle moved from one place at a constant speed, it would move forever and would never take the same linear position again. Trying to map an infinite axis onto a finite circle, we naturally encounter difficulties: if you "roll" your axis onto a circle, each point on the circle will correspond to an infinite number of points on the axis.

So how do we cram infinite into finite? First, let's begin by setting our angular units.

You might be used to expressing angles in degrees. Remember that a degree is an angle which encompasses 1/360 of a full circle. 1/360 is obviously an arbitrary number coming from a historical convention. You could as well split the full circle into 100, 200, 12 or 60 (as in the clock), or 400 parts (1/400 of a full circle is actually a real unit called *grad* or *gradian*), and one would not be much different from all others, apart from the scaling factor. There is a better, more natural unit for measuring angles, the *radian*. One radian is the angle which encompasses the arc length equal to the radius. As you know from middle school, the circumference of a circle equals 2π times its radius, so the full circle encompasses 2π radians. As the radian is the ratio of two lengths, it is dimensionless, and its abbreviation *rad* is mostly omitted ("$\varphi = \pi/2$" rather than "$\varphi = \pi/2$ rad"), whereas you must retain the name of other units ("$\varphi = 90^\circ$", not "$\varphi = 90$", if you mean the quarter of a circle).

Now let's try to figure out the correspondence of angular position to linear. When the point moves an angle φ from the starting point, it covers the arc length φR, which is just the linear displacement from the original position. Thus, $x(t) = R\,\varphi(t)$, where $x(t)$ and $\varphi(t)$ are the linear and angular position functions accordingly.

The ambiguity with the same point having different positions may be resolved in the following way. The *angular position function*, $\varphi(t)$, is proportional to the distance covered. For example, the point started at zero and moved three full circles counterclockwise. It covered the distance $6\pi R$; thus, its final angular position is 6π, although this point coincides with the points with angular positions 0, 2π, 4π, …

Having defined the angular position, it is easy to define the rates of change of first and second order, namely first and second time derivatives of it. Taking the definition

$$\phi(t)=\frac{x(t)}{R} \quad ,$$

we have angular speed

$$\omega=\frac{d\phi}{dt}=\frac{1}{R}\frac{dx}{dt}=\frac{v}{R}$$

and angular acceleration

$$\epsilon=\frac{d\omega}{dt}=\frac{a}{R} \quad ,$$

so all angular kinematic quantities are proportional to the corresponding linear ones via the inverse radius of the circle.

Likewise, in certain situations when ε = const, integrating the definitions will give you

$$\begin{cases} \omega=\omega_0+\epsilon t \\ \phi=\omega_0 t+\dfrac{\epsilon t^2}{2} \end{cases} \quad ,$$

which would be appropriate to call the basic rotational kinematic equations.

Problem. A merry-go-round is stationary. A dog is running on the ground just outside its circumference, moving with a constant angular speed of $\omega = 0.750$ rad/s. The dog does not change his pace when he sees what he has been looking for: a bone resting on the merry-go-round one third of a revolution in front of him. At the instant the dog sees the bone (t = 0), the merry-go-round begins to move in the direction the dog is running, with a constant angular acceleration of ε = 0.0165 rad/s^2. 1). At what time will the dog reach the bone? 2). The confused dog keeps running and passes the bone. How long after the merry-go-round starts to turn do the dog and the bone draw even with each other for the second time? The third time? The fourth time?

Solution. The condition "dog meets bone" on a straight line would be very unambiguous: $x_{dog}(t) = x_{bone}(t)$, i.e. if two points meet, they must have the same position. On the circle, as we pointed out above, this does not have to be the case. The angular positions of the two may be equal, but they may differ by any number of the full circles. For example, let's imagine the initial positions of both were 0; the bone kept lying there and the dog made 4 full circles. They are at the same point now (they meet each other again, so to say), but the bone's position is still 0 and the dog's is 8π. Thus, the difference of the dog's and the bone's position must be $2\pi n$, where n is any integer (including zero). Let's write out their position functions:

$$\begin{cases} \phi_{dog}(t)=\omega t \\ \phi_{bone}(t)=\frac{2\pi}{3}+\frac{\epsilon t^2}{2} \end{cases}$$

(the dog's initial position is defined as zero and it does not accelerate, and the bone's initial position will be one third of a full circle or $2\pi/3$ and its acceleration is ε), and then set their difference to $2\pi n$:

$$\frac{\epsilon t^2}{2}+\frac{2\pi}{3}-\omega t=2\pi n$$

(setting the difference $\phi_{dog}-\phi_{bone}$ instead would be equivalent to picking $-n$ instead of n; as n is any integer, it does not matter).

The above equation, solved for time t, will give us the moments when the dog and the bone meet. As it is quadratic, we have two solutions:

$$t_+=\frac{1}{\epsilon}\left(\omega+\sqrt{\omega^2+4\pi\epsilon\left(n-\frac{1}{3}\right)}\right), t_-=\frac{1}{\epsilon}\left(\omega-\sqrt{\omega^2+4\pi\epsilon\left(n-\frac{1}{3}\right)}\right) .$$

Depending on the numeric values of ω and ε, these values may be positive, negative, or complex. In order to find those that correspond to this problem, we need to use the numbers given in the problem statement and run through various values of n. This is best done using spreadsheet software. This is what we have:

ω	**ε**	**n**	**t+**	**t-**
0.7500	0.0165	-3	Err:502	Err:502
		-2	62.4560	28.4531
		-1	77.8683	13.0408
		0	88.0251	2.8840
		1	96.1877	-5.2786
		2	103.2079	-12.2988
		3	109.4627	-18.5536
		4	115.1586	-24.2495

For $n \leq -3$ we obviously have no solution: both square roots have negative values under them. For n=1 and above the second (minus-sign) solution of the quadratic equation gives negative values. Picking all positive values starting from the smallest, we have the following meeting times:

t_1 = 2.9 s
t_2 = 13.0 s
t_3 = 28.5 s
t_4 = 62.5 s
t_5 = 77.9 s
…..........

A curious student would wonder what happens physically when the valid solutions switch from those of t_- to those of t_+, as from 28.5 s to 62.5 s. I leave this point as an exercise.