Momentum

Besides energy, there is one more physical quantity in mechanics that is sometimes conserved (and there is even more, but let's wait). To begin, let us imagine a system of many particles or objects in the widest sense of the word. For example, a box in the car can be a system of two objects "box+car". Fifteen billiard balls on the table can also be a system of 15 objects. We can even consider 7 of them and consider *them* the system "7 billiard balls", and anything besides them will be "other stuff from the outside world". In general, let's have n particles in our system. They can interact with each other and also with the stuff from the outside, for example, with Earth (it attracts each one of them with the force of gravity) or with air (wind can exert a force on all of them or some of them). We will from now on pay particular attention to distinguishing between the forces coming from the outside and those from the inside (that is, from other particles in the system).

Now, let's concentrate our attention on some random particle number i (where $1 \le i \le n$). We can obviously write second Newton's law for it:

$$\vec{F}_{i\,net} = m_i \vec{a}_i$$

The net force on it consists of the forces acting from the outside world, which we will denote $\vec{F}_i$, and the forces acting from other particles in the system. Each particle number j, in general, acts on our particle number i with the force which we shall denote $\vec{f}_{ij}$ (the first index is the number of particle affected, the second is the number of particle acting). As there are n particles in the system, the total force on our i^{th} particle will be the sum of all those $n - 1$ forces:

$$\sum_{j=1}^{n} \vec{f}_{ij}$$

where, of course, $j \neq i$ (the particle cannot exert a force on itself). Therefore, the net force on the i^{th} particle will be

$$\vec{F}_{i\,net} = \vec{F}_i + \sum_{j=1}^{n} \vec{f}_{ij} \quad .$$

Notice that the particles don't have to *actually* interact with one another: we are considering the most general case. Some or all particles may not be involved in interaction, and for those the vectors $\vec{f}_{ij}$ will just be zero.

Plugging it back in the second Newton's law, we have:

$$m_i \vec{a}_i = \vec{F}_i + \sum_{j=1}^{n} \vec{f}_{ij}$$

for our selected particle number i.

Let's now sum up the Newton's equations for all particles in the system:

$$\sum_{i=1}^{n} m_i \vec{a}_i = \sum_{i=1}^{n} \vec{F}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \vec{f}_{ij} \quad \text{, where } j \neq i \text{ .}$$

Let's look closely at the double sum in this equation. It sums up *all* $\vec{f}_{ij}$ in the system – the forces of interaction of each pair of particles. But now it's a good idea to remember third Newton's law, which states that if two objects interact, the forces they exert on each other are equal in magnitude and opposite in direction. In our notation, for any two particles numbered *i* and *j*, it will be $\vec{f}_{ij} = -\vec{f}_{ji}$. That means that for any $\vec{f}_{ij}$ in the double sum, there will be $\vec{f}_{ji}$ as well, and by third Newton's law they will cancel each other (for example, in the sum $\vec{f}_{13,28}$ will cancel $\vec{f}_{28,13}$, $\vec{f}_{4,9}$ will cancel $\vec{f}_{9,4}$, etc.). It's now easy to see that all $\vec{f}_{ij}$ will cancel, and thus, the double sum equals zero!

We are now left with this:

$$\sum_{i=1}^{n} m_i \vec{a}_i = \sum_{i=1}^{n} \vec{F}_i \quad .$$

Let's now express the acceleration as the derivative of velocity and integrate it:

$$\sum_{i=1}^{n} m_i \frac{d\vec{v}_i}{dt} = \sum_{i=1}^{n} \vec{F}_i \quad , \quad \frac{d}{dt} \sum_{i=1}^{n} m_i \vec{v}_i = \sum_{i=1}^{n} \vec{F}_i \quad , \quad \sum_{i=1}^{n} m_i \vec{v}_i = \int \sum_{i=1}^{n} \vec{F}_i dt = \int \vec{F}_{ext} dt \quad ,$$

where $\vec{F}_{ext} = \sum_{i=1}^{n} F_i$ is the <u>total external force</u> (sum of all the forces from the outside world) acting on our system of particles. Let's again look at the conclusion we have arrived at:

$$\sum_{i=1}^{n} m_i \vec{v}_i = \int \vec{F}_{ext} dt \quad .$$

Now we have to define the new physical quantities we see here. The product of mass and velocity is called **momentum**. The momentum of the i^{th} particle is $\vec{p}_i = m_i \vec{v}_i$. It is a *vector quantity*. Another quantity is the integral of the force by time on the right-hand side. It is called the **impulse** of the force. Thus, the change of the total momentum of the system of particles equals the impulse of the *external* force.

Particularly, the most useful result is obtained in the case when the external force is absent, i.e., equals zero. Since the integral of zero is a constant, the sum of all momenta of the particles will stay constant. This is the law of **conservation of momentum**:

$$\sum_{i=1}^{n} \vec{p}_i = const \quad \text{if} \quad \vec{F}_{ext} = 0 \quad .$$

Since momentum if a vector quantity, its conservation will give you up to three scalar equations of conservation of its components on all three axes. In most of our problems there will be one or two. Also, sometimes the external force does exist, but its component may be zero in a certain direction; then the momentum's component in this direction will be conserved (see the problem below).

Problem. Two particles of masses m_1 and m_2 are moving without friction along a straight horizontal line toward each other with speeds v_1 and v_2 and collide head-on. Find their speeds after the collision in the two following scenarios: 1) they stick together (inelastic collision); 2) no loss of energy (elastic collision).

Solution. Consider the system of these two particles. Is there an outside force acting on the system? On Earth, this will be gravity, but in the horizontal direction its component is zero, so the horizontal momentum of our system is conserved.

Scenario 1.

Before the collision, the momentum of our system is

$$P_1 = m_1 v_1 - m_2 v_2$$

(I chose the positive direction parallel to v_1). After the collision, they stick together and move as one object of mass $m_1 + m_2$ and the speed v':

$$P_2 = (m_1 + m_2) v' \quad .$$

(I assumed the positive direction of v' the same as v_1). The conservation of momentum gives us $P_1 = P_2$, and this answers this question:

$$m_1 v_1 - m_2 v_2 = (m_1 + m_2) v' \quad , \quad v' = \frac{m_1 v_1 - m_2 v_2}{m_1 + m_2} \quad .$$

Scenario 2. Here, the particles move separately, and energy is conserved, as stated in the problem. Thus, we have two conservation equations – for momentum and energy:

$$\begin{cases} m_1 v_1 - m_2 v_2 = -m_1 v_1' + m_2 v_2' \\ \dfrac{m_1 v_1^2}{2} + \dfrac{m_2 v_2^2}{2} = \dfrac{m_1 v_1'^2}{2} + \dfrac{m_2 v_2'^2}{2} \end{cases} \quad .$$

Here I assumed that each particle changes the direction of speed (if not, the speed after the collision, which is denoted with a prime, will be negative). Also, there is no change in potential energy, only kinetic energy participates. This is a system of two equations with two unknowns (primed speeds), but one of them is quadratic with respect to them. The straightforward substitution will be cumbersome. However, there is a known trick which allows for easier solving of this system of equations typical of collision problems. Gather all terms pertaining to particle 1 on the left and particle 2 on the right (and cancel the one-half in the energy equation on the way):

$$\begin{cases} m_1 v_1 + m_1 v_1' = m_2 v_2 + m_2 v_2' \\ m_1 v_1^2 - m_1 v_1'^2 = m_2 v_2'^2 - m_2 v_2^2 \end{cases} \text{, so} \quad \begin{cases} m_1 (v_1 + v_1') = m_2 (v_2 + v_2') \\ m_1 (v_1^2 - v_1'^2) = m_2 (v_2'^2 - v_2^2) \end{cases} \quad ,$$

and then divide the second by the first:

$\frac{v_1^2-v_1'^2}{v_1+v_1'}=\frac{v_2'^2-v_2^2}{v_2'+v_2}$, which is just $v_1-v_1'=v_2'-v_2$. Now we have a system of two *linear* (no more quadratic) equations which is trivial:

$$\begin{cases} m_1 v_1 - m_2 v_2 = -m_1 v_1' + m_2 v_2' \\ v_1 - v_1' = v_2' - v_2 \end{cases} .$$

The solution is

$$\begin{cases} v_1' = \dfrac{2 m_2 v_2 + (m_2 - m_1) v_1}{m_1 + m_2} \\ v_2' = \dfrac{2 m_1 v_1 - (m_2 - m_1) v_2}{m_1 + m_2} \end{cases} .$$

For some Physics 1600 students especially: do not try to copy or memorize these answers! The plus and minus signs at many places here strongly depend on the initial assumptions. For example, it would be less confusing to choose one positive direction for the whole setup, and just add the momenta without worrying about plus or minus signs, but in this case one of the initial speeds would have to be negative since they move toward each other. In this case, the answers would have different plus and minus signs at some places. You must be able to do the solution for this kind of problems yourself.