Two-Dimensional Motion

Thanks to the nature of the Cartesian coordinate system, a particle's motion in two dimensions may be considered as a superposition of two independent one-dimensional motions along either axis. Most of two-dimensional motion problems are about free fall, when the acceleration equals g (9.8 m/s^2 on Earth) and is directed downwards. If you direct one of the axes (say, x) horizontally, the projection of g on it will be zero, thus making the motion along this axis to be uniform, not accelerated (see Fig. 1).

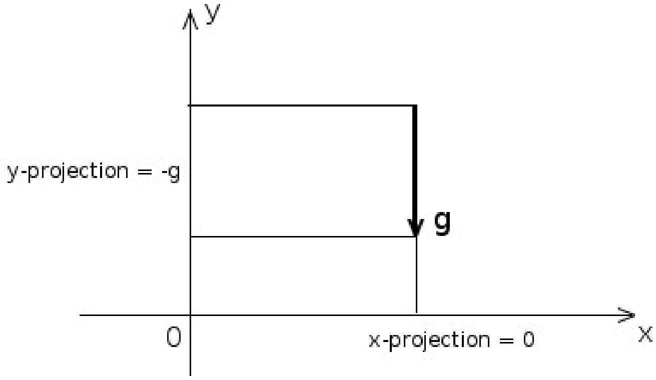

Fig. 1

Of course, you can position your coordinate system in whatever way you like. Actually, in Newton's second law problems, it is convenient to always position one of the axes along the vector of acceleration and in many setups this would mean under an angle. **Not here:** see what happens if the axes are not horizontal or vertical (Fig. 2):

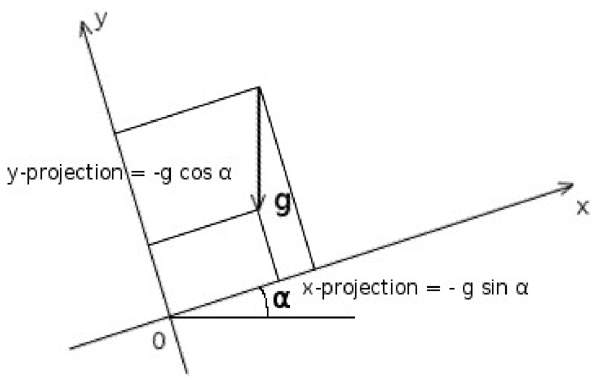

Fig. 2

As the vector of g is still directed downwards (planet Earth doesn't care about your choice of axes), you have accelerated motion on both axes now (neither axis has zero acceleration)! Of course, it won't affect your solution and answer if you work the problem in these coordinates, it will just add more unnecessary calculations.

Most two-dimensional motion problems resemble each other and can be solved in the following four steps.

1. **Introduce the coordinate axes.** The most convenient choice would be to position x-axis horizontally, y-axis vertically, and the origin at the place where the particle starts moving.

2. **Write out the position functions** (now you have two one-dimensional motions, x(t) and y(t), which can also be considered as a <u>vector position function</u> $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = x(t)\,\vec{i} + y(t)\,\vec{j}$, if you want):

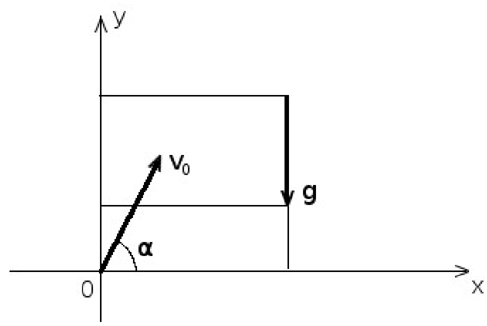

Fig. 3

The one-dimensional generic position function of accelerated motion is $x(t) = x_0 + v_0 t + \frac{a t^2}{2}$.
In our case, the initial positions x_0 and y_0 are zero because we were smart enough to put the origin in the beginning point. The initial speeds on each axis are the projections of the initial velocity vector. If its magnitude is v_0 and the angle with the x-asis is α, then the x-projection of velocity is $v_0 \cos \alpha$ and its y-projection is $v_0 \sin \alpha$. Finally, the acceleration in our coordinates is zero on x-axis and $-g$ on y-axis. Using all this, we write the position functions:

$$x(t) = v_0 \cos\alpha\, t$$

$$y(t) = v_0 \sin\alpha\, t - \frac{g t^2}{2}$$

3. **Find the "convenient point"** which has certain data known, for example, its coordinates or the moment of time when the particle is at that point, depending on the problem. For example, let's say the particle finished its trip at the time $t=T$ when it was at $x = x_0$ and $y = y_0$. The values of x_0, y_0, and T may or may not be known, depending on the problem. Plug this moment into your position functions:

$$\begin{cases} x_0 = v_0 \cos\alpha\, T \\ y_0 = v_0 \sin\alpha\, T - \frac{g T^2}{2} \end{cases}$$

Instead of *functions*, you now have the two *equations* consisting only of *constants*, and hopefully with two unknowns. (If there are more than two unknowns, check for other ways to find them or blame the author who could possibly have given you a problem with insufficient given data.) Now you just have to solve this system for the required unknowns.

4. **Solve the system of equations for the required unknowns.**

That's all about it. There are more interesting two-dimensional motion problems, for example, with multiple objects or when one of the objects is not free-falling but goes along another trajectory, but most of the Physics 1600 2-D problems are identical.

Example problem 1. You can throw a stone with the initial speed v_0 = 10 m/s in any direction on a plane surface. Under what angle do you have to throw it to achieve the farthest range, and what is this range? Neglect the initial height (that is, assume that the stone starts from the ground level).

Solution. Let's first find out the dependence of the range on the angle of the initial velocity vector with the horizontal, and then try to seek the maximum of this function. Choosing the coordinates in our usual way (x horizontal, y vertical, origin at the beginning), the position functions are

$$\begin{cases} x(t)=v_0\cos\alpha\, t \\ y(t)=v_0\sin\alpha\, t-\dfrac{g t^2}{2} \end{cases}$$

The convenient point is of course the place where the stone falls. The coordinates of it are $x=\ell$ (the range) and $y=0$ (the ground level); we called the range ℓ. Calling the time of the flight T, we get two equations:

$$\begin{cases} \ell=v_0\cos\alpha\, T \\ 0=v_0\sin\alpha\, T-\dfrac{g T^2}{2} \end{cases}$$

From the second equation, canceling one of the Ts (knowing that it is certainly not zero) we solve for T:

$$T=\frac{2v_0\sin\alpha}{g}$$

Plugging in into the first equation, we get ℓ as a function of α:

$$\ell=v_0\cos\alpha\frac{2v_0\sin\alpha}{g}=\frac{v_0^2}{g}2\sin\alpha\cos\alpha=\frac{v_0^2}{g}\sin 2\alpha$$

As we see, we don't even have to use calculus to find the maximum of the function $\ell(\alpha)$. The value of sine cannot exceed 1, so the maximum range is $\dfrac{v_0^2}{g}$, which in this problem will be 10.2 m, and the sine becomes 1 when $2\alpha = 90^\circ$, so $\alpha = 45^\circ$.

Example problem 2. Prove that the trajectory of a free-falling object moving in two dimensions is a parabola.

Solution. The trajectory is the graph of one coordinate against the other. With x horizontal and y vertical, we have to have something like $y(x)= a\, x^2 + b\, x + c$, and if y is upwards, a must be negative (from common experience we know that the parabola must be ends-down). All we have to do is write the position functions and exclude the time variable, thus making one equation with x's and y's:

$$\begin{cases} x(t)=v_0\cos\alpha\, t \\ y(t)=v_0\sin\alpha\, t-\dfrac{g t^2}{2} \end{cases}$$

Solving the first one for t and substituting into the second one, we get:

$$y=v_0\sin\alpha\frac{x}{v_0\cos\alpha}-\frac{g}{2}\frac{x^2}{v_0^2\cos^2\alpha}=x\tan\alpha-\frac{g}{2v_0^2\cos^2\alpha}x^2 \quad ,$$

so the y-coordinate is quadratic with respect to x (it has the x-term and x^2-term and a zero free term because of the choice of the origin), which means that the trajectory $y(x)$ is a parabola with the ends down (notice that the quadratic term has a negative coefficient).

Note. All of this is true only when the initial speed is much less than the escape velocity of Earth and the heights are very close to the surface of the planet so we can assume that g stays constant and the Earth surface is flat within the range of the observed event. At heights much greater than those of commercial airliners and speeds close to the escape velocity, the trajectories will be elliptic arcs.