PROBLEM SET

of the Russian National Undergraduate Mathematics Olympiad

(III round, 2007)

1. The triangle ABC has the side vectors $\vec{AB}=\vec{a}, \vec{BC}=\vec{b}, \vec{CA}=\vec{c}$. Show that there exists a triangle with the side vectors $\vec{a}+2\vec{b}, \vec{b}+2\vec{c}, \vec{c}+2\vec{a}$ of the area three times as large.
2. The ends of a hypotenuse of a right-angle triangle are sliding along the coordinate axes $(x\geq 0, y\geq 0)$. What curve is the right-angle vertex making?
3. Solve the equation $y\int_0^x y\,dx-\int_0^x y^2\,dx=y^2-1$
4. Solve the equation $x(y')^2+\sqrt{x}(y^2)'+y^2=0$, $y(1) = 1$
5. Show that the terms of the series $\sum_{n=1}^{\infty} a_n$, where $a_{n+1}=\frac{a_n}{1+a_n}$ $(a_1>0)$, approach zero, but the series is divergent.
6. In a uniform circular cone the height is four times as large as the radius of the base. What angle with the vertical will the axis of the cone make if it is suspended by a point on the circumference of the base?
7. Find the maximum of the areas of the triangles whose vertices lie on the concentric circles of radii 7 and 15. Assume that two of the vertices lie on the greater circle.
8. Prove the inequality $\sum_{n=0}^{\infty} q^{2n+1}\frac{\sin(2n+1)x}{2n+1}\leq \operatorname{arctg} q$ for $0\leq q<1$.
9. Check that the determinant $\begin{vmatrix} \vec{a}\vec{a} & \vec{a}\vec{b} & \vec{a}\vec{c} \\ \vec{b}\vec{a} & \vec{b}\vec{b} & \vec{b}\vec{c} \\ \vec{c}\vec{a} & \vec{c}\vec{b} & \vec{c}\vec{c} \end{vmatrix}$, consisting of scalar products, equals the volume squared of the parallelepiped built on the vectors $\vec{a}, \vec{b}, \vec{c}$.