PROBLEM SET
of the Russian National Undergraduate Mathematics Olympiad
2006

1. Invert an $n \times n$ matrix whose main diagonal elements are x and all other are 1.
2. Find all solutions of the equation $\vec{a} \times (\vec{a} \times (\ldots(\vec{a} \times \vec{x})\ldots)) = \vec{x}$. The non-zero vector $\vec{a}$ in the left-hand part occurs n times.
3. Write the equation of a circular cone, for which the lines $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and $\frac{x}{3} = \frac{y-1}{1} = \frac{z-2}{2}$ are its generatrices and the axis of symmetry of the cone lies in the plane of these lines.
4. The points *A, B, C* of a triangle are at the distances 7, 15, 15 from the origin. What is the maximum value of the area of the triangle *ABC*?
5. Find $\lim\limits_{x \to a} \frac{x^x - x^a}{x^x - a^x}$.
6. Check that $\int \frac{x^3 - y^3}{x + y^2} dx = \frac{1}{3}(x^3 + y^3) + C$. Here y is the function given implicitly by the equation $x^3 + 3xy + y^3 = 1$.
7. What can the function $q(x) \neq 0$ be if the equation $y'' = (\lambda^2 + q(x))y$ has the solution $y = e^{\lambda x}(1 + \frac{A(x)}{\lambda})$ for any $\lambda \neq 0$?
8. Find $\int_0^1 F(x)\,dx$, where $F(x) = \int_x^1 \frac{1}{t^2+1} \cdot \frac{1}{(t-x)^2+1} dt, (0 \leq x \leq 1)$.
9. The function $f(x)$ is defined by the series $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Show that $f(x) + f(1-x) = f(1) - \ln x \ln(1-x)$.
10. The elements of a 2×2 matrix are randomly selected from the interval [0,1]. Find the dispersion of the determinant of this matrix.